A Lower Bound on the Interactive Capacity of Binary Memoryless Symmetric Channels

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Abstract

The interactive capacity of a channel is defined in this paper as the maximal rate at which the transcript of any interactive protocol can be reliably simulated over the channel. It is shown that the interactive capacity of any binary memoryless symmetric (BMS) channel is at least $0.0302$ its Shannon capacity. To that end, a rewind-if-error coding scheme for the simpler binary symmetric channel (BSC) is presented, achieving the lower bound for any crossover probability. The scheme is based on extended-Hamming codes combined with randomized error detection. The bound is then shown to hold for any BMS channel using extremes of the Bhattacharyya parameter. Finally, it is shown that the public randomness required for error detection can be reduced to private randomness in a standard fashion, and can be extracted from the channel without affecting the overall asymptotic rate. This gives rise to a fully deterministic interactive coding scheme achieving our lower bound over any BMS channel.

I. INTRODUCTION

In the classical Shannon one-way communication problem, a transmitter (Alice) wishes to send a message reliably to a receiver (Bob) over a memoryless noisy channel. She does so by mapping her message into a sequence of channel inputs (codeword) in a predetermined way, which is corrupted by the channel and then observed by Bob, who tries to recover the original message. The Shannon capacity of the channel, which is the maximal number of message bits per channel use that Alice can convey to Bob with vanishingly low error probability, quantifies the most efficient way to do so. In the two-way channel setup [1], both parties draw independent messages and wish to exchange them over a two-input two-output memoryless noisy channel, and the Shannon capacity (region) is defined similarly. Unlike the one-way case, both parties can now employ adaptive coding by incorporating their respective observations of the past channel outputs into their transmission processes. However, just as in the one-way setup, the messages they wish to exchange are determined before communication begins. In other words, if Alice and Bob had been connected by a noiseless bit pipe, they could have simply sent their messages without any regard to the message of their counterpart.

In a different two-way communication setup, generally referred to as interactive communication, the latter assumption is no longer held true. In this interactive communication setup, Alice and Bob do not necessarily wish to disclose all their local information. What they want to tell each other depends, just like in human conversation, on what the other would tell them. A simple instructive example (taken from [2]) is the following. Suppose that Alice and Bob play chess remotely, by announcing their moves over a communication channel (using, say, 12 bits per move, which is clearly sufficient). If the moves are conveyed without error, then both parties can keep track of the state of the board, and the game can

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proceed to its termination. The sequence of moves occurring over the course of this noiseless game is called a transcript, and it is dictated by the protocol of the game, which constitutes Alice and Bob’s respective strategies determining their moves at any given state of the board.

Now, assume that Alice and Bob play the game over a noisy two-way channel, yet wish to simulate the transcript as if no noises were present. In other words, they would like to communicate back and forth in a way that ensures, once communication is over, that the transcript of the noiseless game can be reproduced by both parties with a small error probability. They would also like to achieve this goal as efficiently as possible, i.e., with the least number of channel uses. One direct way to achieve this is by having both parties describe their entire protocol to their counterpart, i.e., each and every move they might take given each and every possible state of the board. This reduces the interactive problem to a non-interactive one, with the protocol becoming a pair of messages to be exchanged. However, this solution is grossly inefficient; the parties now know much more than they really need in order to simply reconstruct the transcript. At the other extreme, Alice and Bob may choose to describe the transcript itself by encoding each move separately on the fly, using a short error correcting code. Unfortunately, this code must have some fixed error probability and hence an undetected error is bound to occur at some unknown point, causing the states of the board held by the two parties to diverge, and rendering the remainder of the game useless. It is important to note that if Alice and Bob had wanted to play sufficiently many games in parallel, then they could have used a long error-correcting code to simultaneously protect the set of all moves taken at each time point, in which principle would have let them operate at the one-way Shannon capacity (which is the best possible). The crux of the matter therefore lies in the fact that the interactive problem is one-shot, namely, only a single instance of the game is being played.

In light of the above, it is perhaps surprising that it is nevertheless possible to simulate any one-shot interactive protocol using a number of channel uses that is proportional to the length of the transcript, or in other words, that there is a positive interactive capacity whenever the Shannon capacity is positive. This fact was originally proved by Schulman [3], who was also the first to introduce the notion of interactive communication over noisy channels. However, the interactive capacity has never been quantified; it is only known to be some nonzero fraction of the Shannon capacity. In this paper, we show that for a large class of channels, the interactive capacity is at least 0.0302 of the Shannon capacity.

The rest of the paper is organized as follows. In Section II we present the problem formulation and a high level description of the techniques. In Section III we put our work in context of existing results in the literature. We provide some necessary preliminaries in Section IV and then state the main results in Section V. The coding scheme used in the proof for the binary symmetric channel (BSC) is presented and analyzed in Sections VI and VII respectively, and then generalized to binary memoryless symmetric (BMS) channels in Section VIII. Finally, in Section IX, we explain how the randomized coding scheme can be modified to be fully deterministic.

II. PROBLEM FORMULATION AND THE MAIN CONTRIBUTION

In this paper, a length-n interactive protocol is the triplet \( \pi \triangleq (\phi_{\text{Alice}}, \phi_{\text{Bob}}, \psi) \), where

\[
\phi_{\text{Alice}} \triangleq \{ \phi_{\text{Alice}}^i : \{0, 1\}^{i-1} \rightarrow \{0, 1\} \}_{i=1}^n \\
\phi_{\text{Bob}} \triangleq \{ \phi_{\text{Bob}}^i : \{0, 1\}^{i-1} \rightarrow \{0, 1\} \}_{i=1}^n \\
\psi \triangleq \{ \psi_i : \{0, 1\}^{i-1} \rightarrow \{\text{Alice, Bob}\} \}_{i=1}^n.
\]

The functions \( \phi_{\text{Alice}} \) are known only to Alice, and the functions \( \phi_{\text{Bob}} \) are known only to Bob. The speaker order functions \( \psi \) are known to both parties. The transcript \( \tau \) associated with the protocol \( \pi \) is
sequentially generated by Alice and Bob as follows

$$\tau_i = \begin{cases} \phi_A^{\text{Alice}}(\tau_{i-1}) & \sigma_i = \text{Alice} \\ \phi_B^{\text{Bob}}(\tau_{i-1}) & \sigma_i = \text{Bob} \end{cases}$$ (1)

where $\sigma_i$ is the identity of the speaker at time $i$, which is given by:

$$\sigma_i = \psi_i(\tau_{i-1}).$$ (2)

In the interactive simulation problem Alice and Bob would like to simulate the transcript $\tau$, by communicating back and forth over a noisy memoryless channel $P_{Y|X}$. Specifically, we restrict our discussion to channels with a binary input alphabet $X = \{0, 1\}$, and a general (possibly continuous) output alphabet $Y$. Note that while the order of speakers in the interactive protocol itself might be determined on the fly (by the sequence of functions $\psi$), we restrict the simulating protocol to use a predetermined order of speakers, due to the fact that our physical channel model does not allow simultaneous transmissions.

To achieve their goal, Alice and Bob employ a length-$N$ coding scheme $\Sigma$ that uses the channel $N$ times. The coding scheme consists of a disjoint partition $\hat{A} \sqcup \hat{B} = \{1, \ldots, N\}$ where $\hat{A}$ (resp. $\hat{B}$) is the set of time indices where Alice (resp. Bob) speaks. This disjoint partition can be a function of $\psi$, but not of $\phi_A^{\text{Alice}}, \phi_B^{\text{Bob}}$. At time $j \in \hat{A}$ (resp. $j \in \hat{B}$), Alice (resp. Bob) sends some Boolean function of $(\phi_A^{\text{Alice}}, \psi)$ (resp. $(\phi_B^{\text{Bob}}, \psi)$), and of everything she has received so far from her counterpart. The rate of the scheme is $R = \frac{N}{n}$ bits per channel use. When communication terminates, Alice and Bob produce their simulations of the transcript $\tau$, denoted by $\hat{\tau}_A(\Sigma, \phi_A^{\text{Alice}}, \psi) \in \{0, 1\}^n$ and $\hat{\tau}_B(\Sigma, \phi_B^{\text{Bob}}, \psi) \in \{0, 1\}^n$ respectively. The error probability attained by the coding scheme is the probability that either of these simulations is incorrect, i.e.,

$$P_e(\Sigma, \pi) \triangleq \Pr(\hat{\tau}_A(\Sigma, \phi_A^{\text{Alice}}, \psi) \neq \tau \lor \hat{\tau}_B(\Sigma, \phi_B^{\text{Bob}}, \psi) \neq \tau).$$

A rate $R$ is called achievable if there exists a sequence $\Sigma_n$ of length-$N_n$ coding schemes with rates $\frac{n}{N_n} \geq R$, such that

$$\lim_{n \to \infty} \max_{\pi \text{ of length } n} P_e(\Sigma_n, \pi) = 0,$$

where the maximum is taken over all length-$n$ interactive protocols. Accordingly, we define the interactive capacity $C_I(P_{Y|X})$ as the maximum of all achievable rates for the channel $P_{Y|X}$. Note that this definition parallels the definition of maximal error capacity in the one-way setting, as we require the error probability attained by the sequence of coding schemes to be upper bounded by a vanishing term uniformly for all protocols.

It is clear that at least $n$ bits need to be exchanged in order to reliably simulate a general protocol, hence the interactive capacity satisfies $C_I(P_{Y|X}) \leq 1$. In the special case of a noiseless channel, i.e., where the output deterministically reveals the input bit, and assuming that the order of speakers is predetermined (namely $\psi$ contains only constant functions), this upper bound can be trivially achieved; Alice and Bob can simply evaluate and send $\tau_i$ sequentially according to (1) and (2). Note however, that if the order of speakers is general, then this is not a valid solution, since we required the order of speakers in the coding scheme to be fixed in advance. Nevertheless, any general interactive protocol can be sequentially simulated using the channel $2n$ times with alternating order of speakers, where each party sends a dummy bit whenever it is not their time to speak. Conversely, a factor two blow-up in the protocol length in order to account for a non pre-determined order of speakers is also necessary. To see this, consider an example of a protocol where Alice’s first bit determines the identity of the speaker for the rest of time; in order to simulate this protocol using a predetermined order of speakers, it is easy to see that at least
$n - 1$ channel uses must be allocated to each party in advance. We conclude that under our restricting capacity definition, the interactive capacity of a noiseless channel is exactly $\frac{1}{2}$.

When the channel is noisy, a tighter trivial upper bound holds:

$$C_I(P_{Y|X}) \leq \frac{1}{2} C_{Sh}(P_{Y|X}),$$

(3)

where $C_{Sh}(P_{Y|X})$ is the Shannon capacity of the channel. To see this, consider the same example given above, and note that each party must have sufficient time to reliably send $n - 1$ bits over the noisy channel. Hence, the problem reduces to a pair of one-way communication problems, in which the Shannon capacity is the fundamental limit. We remark that it is reasonable to expect the bound (3) to be loose, since general interactive protocols cannot be trivially reduced to one-way communication as the parties cannot generate their part of the transcript without any interaction. However, the tightness of the bound remains a wide open question. We note in passing that if we had considered simulating only protocols with a predetermined order of speakers, the corresponding upper bound would have been $C_I(P_{Y|X}) \leq C_{Sh}(P_{Y|X})$.

**Channel Models and Capacity Lower Bounds**

The first noisy channel model we consider is the memoryless binary symmetric channel with crossover probability $0 \leq \varepsilon \leq \frac{1}{2}$, BSC($\varepsilon$). The input to output relation of the BSC($\varepsilon$) is given by

$$Y = X \oplus Z$$

where $X, Y, Z \in \mathbb{F}_2$, $\oplus$ denotes addition over $\mathbb{F}_2$. $Z$ is statistically independent of $X$ with $\Pr(Z = 1) = \varepsilon$. We denote its Shannon capacity by

$$C_{Sh}(\varepsilon) \equiv 1 - h(\varepsilon),$$

where $h(\varepsilon) \equiv -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$ is the binary entropy function, and $\log(x) \equiv \log_2(x)$. We also use $C_I(\varepsilon)$ to denote the interactive capacity of the BSC($\varepsilon$).

A richer channel model which is commonly used in the coding literature is the binary memoryless symmetric (BMS) channel [4]–[8]. While several equivalent definitions exist, we choose to define a BMS channel as a collection of BSC with various crossover probabilities [9] as follows:

**Definition 1. [BMS channels]** A memoryless channel with binary input $X$ output $Y$ and a conditional distributions $P_{Y|X}$ is called binary memoryless symmetric channel (BMS($P_{Y|X}$)) if there exists a sufficient statistic of $Y$ for $X$: $g(Y) = (X \oplus Z_T, T)$ , where $(T, Z_T)$ are statistically independent of $X$, $Z_T$ is a binary random variable with $\Pr(Z_T = 1|T = t) = t$, and $0 \leq T \leq \frac{1}{2}$ with probability one.

Consequently, the Shannon capacity of BMS($P_{Y|X}$) channel is

$$C_{Sh}(P_{Y|X}) = 1 - \mathbb{E}h(T).$$

Important BMS channels other than the BSC, include the binary erasure channel (BEC), the binary additive white Gaussian noise (BiAWGN) among others, as elaborated in Section VIII.

The main contribution of this paper is the following bound for the ratio between the interactive capacity and the Shannon capacity for any BMS channel.
Theorem 1. For any BMS($P_{Y|X}$) channel with positive Shannon capacity $C_{Sh}(P_{Y|X})$ and interactive capacity $C_{I}(P_{Y|X})$

\[
\frac{C_{I}(P_{Y|X})}{C_{Sh}(P_{Y|X})} \geq 0.0302.
\]

Theorem 1 is proved by first analyzing the BSC special case stated in the following theorem, and then extending the result for a general BMS channel.

Theorem 2. For any BSC with crossover probability $0 \leq \varepsilon \leq 1/2$, Shannon capacity $C_{Sh}(\varepsilon)$ the and interactive capacity $C_{I}(\varepsilon)$ the following bound holds:

\[
\frac{C_{I}(\varepsilon)}{C_{Sh}(\varepsilon)} \geq 0.0302.
\]

The first step in the proof is standardly symmetrizing the order of speakers by possibly adding dummy transmissions, such that Alice speaks at odd times, and Bob speaks at even times. In the sequel we refer to this order of speakers as bit-vs.-bit. This reduces the rate by a factor of two at most. Theorem 2 is then proved by using a rewind-if-error scheme in the spirit of [3], [10] designed for simulating the transcript of protocols with an alternating order of speakers. As mentioned in the chess game example, in the general case, the transcript bits of an interactive protocol should be decoded instantaneously, which implies that error correction codes (that typically use long blocks) cannot be straightforwardly used. Instead, rewind-if-error scheme are based on unencoded transmission followed by error detection and retransmission. Namely, the transcript is simulated in blocks, assuming no errors are present. Then, an error detection phase takes place, initiating the retransmission of the block whenever errors are detected. Since the probability of error in a block increases with the block length, such schemes assume that the channel is almost error free, namely, that $\varepsilon$ is very small compared to the reciprocal of the block length.

The scheme presented in Sections VI and VII of this paper is based on a layered error detection and retransmission. The error detection is implemented by an extended-Hamming code in the first layer, and by a standard randomized error detection algorithm [11] at higher layers. As will be shown in the sequel, the total rate of the proposed scheme is mostly effected by the efficiency of the error detection in the first layer. For this reason, for the first layer the error detection is performed using an extended-Hamming code which is known to be highly efficient for errors generated by a BSC [12].

We analyze the rate of the scheme for a fixed small $\varepsilon$. For BSC with larger values of $\varepsilon$, we reduce the crossover probability via repetition coding and account for the incurred rate loss. The result is then generalized in Section VIII to the case of general BMS channel recalling that the BSC has the largest Bhattacharyya parameter among all the BMS channels with the same Shannon capacity [8]. This property implies that the BSC requires the largest number of repetitions per target crossover probability among all BMS channels with the same capacity and can therefore be regarded as the worst case BMS channel with a given capacity, for the proposed scheme.

The requirements for randomness in the scheme are discussed in Section IX. For simplicity of exposition, the scheme described in Section VII uses randomness for the error detection. In Section IX we show that the requirement for randomness can be circumvented. This is done by first reducing the number of required random bits to $o(n)$ and then standardly extracting them from the noisy channels [13], [14] without reducing the overall rate of the scheme.

III. CONNECTIONS TO THE EXISTING WORK

The interactive communication problem introduced by Schulman [5], [15] is motivated by Yao’s communication complexity scenario [16]. In this scenario, the input of a function $f$ is distributed between
Alice and Bob, who wish to compute \( f \) with negligible error by exchanging (noiseless) bits using some interactive protocol. The length of the shortest protocol achieving this is called the communication complexity of \( f \), and denoted by \( CC(f) \). In the interactive communication setup, Alice and Bob must achieve their goal by communicating through a pair of independent BSC(\( \varepsilon \)).

The minimal length of an interactive protocol attaining this goal is now denoted by \( CC_\varepsilon(f) \).

In [10], Kol and Raz defined the interactive capacity as

\[
C_1^{KR}(\varepsilon) \triangleq \lim_{n \to \infty} \min_{f:CC(f)=n} \frac{n}{CC_\varepsilon(f)},
\]

and proved that

\[
C_1^{KR}(\varepsilon) \geq 1 - O(\sqrt{h(\varepsilon)})
\]

in the limit of \( \varepsilon \to 0 \), under the additional assumption that the communication complexity of \( f \) is computed with the restriction that the order of speakers is predetermined and has some fixed period. The former assumption on the order of speakers is important. Indeed, consider again the example where the function \( f \) is either Alice’s input or Bob’s input as decided by Alice. In this case, the communication complexity with a predetermined order of speakers is double that without this restriction, and hence considering such protocols renders \( C_1^{KR}(\varepsilon) \leq \frac{1}{2} \). For further discussion on speaking order impact as well as channel models that allow collisions, see [17]. For a fixed nonzero \( \varepsilon \), the coding scheme presented in [3] (which precedes [10]) already showed that \( C_1^{KR}(\varepsilon) = \Theta(C_{Sh}(\varepsilon)) \), but the constant has not been computed (and to the best of our knowledge, has not been computed for any scheme hitherto). Both [3] and [10] based their proofs on rewind-if-error coding schemes, i.e., schemes based on a hierarchical and layered error detection and appropriate retransmissions, which is also the approach we take in this paper. Note that our definition of the BSC interactive capacity is stricter than (4), at least in principle, as it requires reconstruction of the entire transcript. For this reason, \( C_1(\varepsilon) \leq C_1^{KR}(\varepsilon) \), hence our lower bound applies to \( C_1^{KR}(\varepsilon) \) as well (and also achieves the asymptotic behavior (4) when simulating bit-vs.-bit protocols). Our capacity definition further enjoys the property of being decoupled from any source coding problem such as function communication complexity.

Another aspect we would like to discuss is the type of randomness used by a coding scheme, which can be either public, private or none. The scheme in [3] requires only private randomness, while [10] requires public randomness. It is interesting to note that Schulman’s tree code scheme [15] is not randomized. However, it is not designated to be rate-wise efficient, and does not achieve the lower bound in (5). A non-random coding scheme was recently proposed by Gelles et. al. [18] which is based on a concatenation of a de-randomized interactive coding scheme and a tree-code.

The rewind-if-error scheme presented in this paper is inspired by the scheme in [10] but its error detection is not based on random hashes but rather on extended-Hamming codes and randomized (yet structured) error detection. Our deterministic coding scheme presented in Section 10X is not based on de-randomization of a randomized coding scheme as [18] but rather on adapting the error detection so it requires a relatively small number of random bits and then standardly extracting them from the noisy channels at hand.

To summarize the discussion above, there are various setups one may consider in interactive communication. Our scheme, and its corresponding lower bound, are based on the most restrictive set of assumptions: the order of speakers can be adaptive in the simulated protocol but is predetermined in the simulating protocol, no private or public randomness are allowed, and the entire transcript must be reconstructed by both parties. Thus, our capacity lower bounds remain valid for any other set of standard assumptions.

The current paper extends the preliminary results presented in [19] in the following aspects: i) The
error detection in the scheme is structured and is not based on random hashes. ii) The rate of the resulting scheme is improved and consequently the lower bound for the ratio between the interactive capacity and Shannon’s capacity is also improved. iii) The scheme described in this paper can be modified to operate on private randomness, which can be fully extracted from the channels and not on public randomness. iv) The results are generalized for BMS channels.

IV. PRELIMINARIES

Let \( D(P||Q) \triangleq \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)} \) denote the Kullback-Leibler Divergence between the distributions \( P(\cdot) \) and \( Q(\cdot) \). Let \( d(p||q) \triangleq p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q} \) denote the Kullback-Leibler Divergence between two Bernoulli random variables with probabilities \( p \) and \( q \). In the sequel we use \( \mathbb{1}(\cdot) \) to denote the indicator function, which equals one if the condition is satisfied and zero otherwise.

The following simple results are used throughout the paper:

**Lemma 1** (Repetition coding over BSC). Let a bit be sent over BSC(\( \varepsilon \)) using \( \rho \) repetitions and decoded by a majority vote (if \( \rho \) is even, ties are broken by tossing a fair coin). The decoding error probability \( P_e \) can be upper bounded by

\[
P_e \leq \beta^\rho = 2^{-\rho d(\frac{1}{2}||\varepsilon)},
\]

where \( \beta \triangleq 2\sqrt{\varepsilon(1-\varepsilon)} \) is the Bhattacharyya parameter respective to the BSC(\( \varepsilon \)). The induced channel from the input bit to its decoded value is thus a BSC(\( P_e \)). The proof is standard (see for example [5]) and can be regarded as special case of Lemma 8 stated and proved in Section VIII. Note that the random tie breaking is done in order to simplify the scheme and its analysis. It does, however, assume private randomness at both parties. In Section IX we show how the random tie breaking can be circumvented.

We now introduce two error detection methods that would be used in the coding scheme. The first one assumes the error are generated by BSC’s and is based on error correction codes:

**Definition 2** (Error detection using an extended-Hamming code). Let \( X^A \) and \( X^B \) be binary (row) vectors of length \( k \) held by Alice and Bob respectively. Let \( H \) be the parity check matrix of an extended-Hamming code with parameters \((k, k - \log k - 1, 4)\). Let \( NEQ \) be a variable set to one if the parties decide that \( X^A \neq X^B \) and set to zero otherwise, calculated according to the following algorithm:

1) Alice calculates her syndrome vector \( s^A = X^A H^T \)
2) Bob calculates his syndrome vector \( s^B = X^B H^T \)
3) Alice sends \( s^A \) (1 + \( \log k \) bits) to Bob
4) Bob calculates \( NEQ = \mathbb{1}(s^A \neq s^B) \)
5) Bob sends \( NEQ \) (1 bit) to Alice

The overall number of bits communicated between Alice and Bob is \( 2 + \log k \).

The performance of this scheme over a BSC(\( \varepsilon \)) is given in the following lemma:

**Lemma 2.** Assume that

\[
X^A = X^B \oplus Z,
\]

where \( Z \) is an i.i.d Bernoulli(\( \varepsilon \)) vector. The probability of a mis-detected error of the scheme in Definition 2 is given by

\[
Pr\left(NEQ = 0, X^A \neq X^B\right) = \frac{1}{2k} \left( 1 + 2(k - 1)(1 - 2\varepsilon)\frac{k}{2} + (1 - 2\varepsilon)^k \right) - (1 - \varepsilon)^k.
\]
The corresponding probability of a false error detection is

\[ \Pr\left( NEQ = 1, X^A = X^B \right) = 0. \]

**Proof.** First, it is clear that for any \( X^A = X^B \) we have \( NEQ = 1, X^A = X^B \) is impossible, so the probability of false error detection is \( \Pr\left( NEQ = 1, X^A = X^B \right) = 0. \) For the probability of error mis-detection, note that \( s^A \oplus s^B = (X^A \oplus X^B)H^T = ZH^T. \) Therefore, the event \( NEQ = 0 \) is identical to the event in which \( s^A \oplus s^B = ZH^T = 0^T, \) i.e., \( Z \) is a codeword in \( H. \) All in all

\[
\Pr\left( NEQ = 0, X^A \neq X^B \right) = \Pr\left( ZH^T = 0^T, Z \neq 0^T \right) = \frac{1}{2k} \left( 1 + 2(k-1)(1-2\varepsilon)^\frac{k}{2} + (1-\varepsilon)^k \right) - (1-\varepsilon)^k,
\]

(7)

where \( \Xi \) is standardly calculated using the dual code [12, p. 52]. \( \square \)

The second error detection scheme is a randomized scheme based on [13, p. 30], which applies for arbitrary vectors:

**Definition 3** (Randomized error detection using polynomials). Let \( X^A \) and \( X^B \) be arbitrary binary vectors of length \( \ell \) held by Alice and Bob respectively. Let \( \gamma \in \mathbb{N} \), where \( \gamma > 1. \) Let \( q \) be a prime number such that \( \gamma \ell \leq q \leq 2\gamma \ell \) (by Bertrand’s postulate such a number must exist). Let \( NEQ^\text{Poly} \) be a variable set to one if the parties decides that \( X^A \neq X^B \) and set to zero otherwise, calculated according to the following algorithm:

1. Alice uniformly draws \( U \in \mathbb{F}_q \)
2. Alice calculates \( A(U, X^A) = \sum_{i=1}^\ell X_i^A U^{i-1} (\text{mod } q) \)
3. Alice sends Bob \( U \) and \( A(U, X^A) \)
4. Bob calculates \( B(U, X^A) = \sum_{i=1}^\ell X_i^B U^{i-1} (\text{mod } q) \)
5. Bob calculates \( NEQ^\text{Poly} = 1 \left( A(U, X^A) - B(U, X^A) \neq 0 \right) \)
6. Bob sends \( NEQ^\text{Poly} \) to Alice

All in all, Alice needs to send at most \( \lceil \log 2\gamma \ell \rceil \) bits for the representation of \( U \), and at most \( \lceil \log 2\gamma \ell \rceil \) bits for the representation of \( A(U, X^A) \). Bob sends Alice one bit.

**Lemma 3.** The error detection scheme of Definition 3 obtains an error mis-detection probability of

\[ \Pr\left( NEQ^\text{Poly} = 0 \mid X^A \neq X^B \right) \leq \frac{1}{\gamma}, \]

and a false error detection probability of

\[ \Pr\left( NEQ^\text{Poly} = 1 \mid X^A = X^B \right) = 0. \]

**Proof.** Note that \( A(U, X^A) \) and \( B(U, X^B) \) are the evaluation at point \( U \) of two polynomials over \( \mathbb{F}_q \) whose (binary) coefficients are the elements of \( X^A \) and \( X^B \) respectively. Clearly, if \( X^A = X^B \), then \( NEQ = 0 \) for every value of \( U \) hence \( \Pr\left( NEQ = 1 \mid X^A = X^B \right) = 0. \) On the other hand, if \( X^A \neq X^B \), \( A(U, X^A) - B(U, X^A) = 0 \) implies that \( U \) is a root of the polynomial

\[ \sum_{i=1}^\ell (X_i^A - X_i^B) U^{i-1} (\text{mod } q). \]

Since the degree of the polynomial is at most \( \ell \), there are at most \( \ell - 1 \) such roots, so

\[ \Pr\left( NEQ = 0 \mid X^A \neq X^B \right) \leq \frac{\ell - 1}{q} < \frac{\ell}{\gamma \ell} = \frac{1}{\gamma}. \]
V. Main Results

We first lower bound $C_1(\varepsilon)$ for $\varepsilon \ll 1$. Our bound is stated in the following theorem:

**Theorem 3.** The transcript of any protocol with $n$ bit-vs.-bit order of speakers (i.e. Alice sends a bit on odd times and Bob sends a bit on even times), can be reliably simulated over $BSC(\varepsilon)$ in the following rate

$$R_{BSC}(\varepsilon, k) \triangleq \frac{1 - k\varepsilon - (3 + \log k)\beta\tilde{a} - \frac{k^2}{k-1} \left( P_{e1} + 3\beta^{a+4}k \log k \cdot \frac{2-\beta^{2}k}{(1-\beta^{2}k)^2} \right) - 3\beta^{a+4}k^2 \log k \cdot \frac{2-\beta^{2}k}{(1-\beta^{2}k)^2}}{1 + \frac{\tilde{a}(3+\log k)}{k} + 3 \log k \left[ \frac{\alpha(2k-1)}{(k-1)^2} + \frac{4k}{(k-1)^3} + \frac{4k-2}{k(k-1)^2} \right]}$$

(8)

where

$$P_{e1} \leq \frac{1}{2k}\left( 1 + 2(k-1)(1-2\varepsilon)^{\frac{\varepsilon}{k}} + (1-2\varepsilon)^k \right) - (1-\varepsilon)^k + (3 + \log k)\beta\tilde{a}. \quad (9)$$

Let $0 < \varepsilon < \frac{1}{16}$, $\beta \triangleq 2\sqrt{\frac{1}{1-\varepsilon}}$, $a = 3$ and $\tilde{a} = 5$. $k$ is can be take as any integer a power of two satisfying $k \leq \frac{a}{\beta}$. 

Using this theorem, $C_1(\varepsilon) \geq \max_k R_{BSC}(\varepsilon, k)$ for protocols with a bit-vs.-bit order of speakers and $C_1(\varepsilon) \geq \frac{1}{2} \max_k R_{BSC}(\varepsilon, k)$ for protocols with a general (possibly adaptive) order of speakers.

The proof of Theorem 3 is by the construction and analysis of a rewind-if-error scheme and appears in Sections VI and VII. We note that the presented scheme is randomized and in Section IX we explain how to modify it to be deterministic.

The following corollary proved in Appendix A states that the scheme obtains the rate lower bound (5) from [10]:

**Corollary 1.** For $\varepsilon \to 0$

$$\max_k R_{BSC}(\varepsilon, k) \geq 1 - O(\sqrt{h(\varepsilon)})$$

As stated before, the presented rewind-if-scheme is designed for BSC with a sufficiently small $\varepsilon$. For larger values of $\varepsilon$, the channel can be converted to a $BSC(\delta')$ with $\delta' \leq \delta < \varepsilon$ using $\rho(\varepsilon, \delta)$ repetitions followed by a majority vote according to Lemma II. The following lemma bounds the interactive capacity by using an interactive coding scheme augmented by a repetition code:

**Lemma 4.** For every $0 < \varepsilon < \frac{1}{2}$ and $0 < \delta < \frac{1}{2}$

$$\frac{C_1(\varepsilon)}{C_{Sh}(\varepsilon)} \geq \frac{C_1(\delta)}{\log \frac{1}{\delta} + 1}.$$

Proof. Let $\rho$ be the smallest integer such that $\beta^\rho \leq \delta$, where $\beta \triangleq 2\sqrt{\frac{1}{1-\varepsilon}}$ is the Bhattacharyya parameter of the $BSC(\varepsilon)$ as above. By Lemma II, using $\rho$ repetitions, the $BSC(\varepsilon)$ can be converted to a $BSC(\delta')$ with $\delta' \leq \delta$. Normalizing by $C_{Sh}(\varepsilon)$ and noting that $C_1(\delta) \leq C_1(\delta')$, we obtain

$$\frac{C_1(\varepsilon)}{C_{Sh}(\varepsilon)} \geq \frac{C_1(\delta)}{\rho(\varepsilon, \delta)C_{Sh}(\varepsilon)}.$$
By Lemma 1

\[ \rho \leq \rho(\varepsilon, \delta) \equiv \frac{1}{\log \frac{1}{\delta}} + 1, \]

where ‘+1’ accounts for rounding to the nearest larger integer. Furthermore,

\[ \rho(\varepsilon, \delta) C_{\text{Sh}}(\varepsilon) = \left( \frac{1}{\log \frac{1}{\delta}} + 1 \right) C_{\text{Sh}}(\varepsilon) \]

\[ \leq \frac{I(X; Y)}{L(X; Y)} \log \frac{1}{\delta} + I(X; Y), \]

(10)

where \( X \sim \text{Bernoulli}(\frac{1}{2}) \) is the input of a BSC(\( \varepsilon \)) channel and \( Y \) is its respective output, \( I(X; Y) = D(P_{XY}||P_X P_Y) = C_{\text{Sh}}(\varepsilon) \) is the mutual information between \( X \) and \( Y \) and \( L(X; Y) = D(P_X P_Y||P_X Y) = D(\frac{1}{2}||\varepsilon) = \log \frac{1}{\delta} \) is the lautum information between \( X \) and \( Y \) \([20]\). Using the facts that for the BSC, \( L(X; Y) \geq I(X; Y) \) \([20]\, \text{Theorem 12} \) and that trivially \( I(X; Y) \leq 1 \), concludes the proof.

Theorem 2 now follows by using Theorem 3 with \( k = 2^9 \) and \( \delta = 0.00018908 \) in order to calculate \( R_{\text{BSC}}(\delta, k) \), dividing the rate by two in order to symmetrize the order of speakers and finally applying Lemma 4.

VI. DESCRIPTION OF THE CODING SCHEME FOR THE BSC

The rewind-if-error scheme is based on two concepts: uncoded transmission and retransmissions based on error detection. The uncoded transmission is motivated by the fact that in a general interactive protocol, even in a noise-free environment, the parties cannot predict the transcript bits to be output by their counterpart, and hence might not always know some of their own future outputs. For this reason, long blocks of bits, which are essential for efficient block codes, cannot be generated.

The concept of retransmissions based on error detection can be viewed as an extension of the classic example of the one-way BEC with feedback \([5, \text{p. 506}] \). In this simple setup, channel errors occur independently with probability \( \varepsilon \) and errors are detected and marked as erasures, whose locations are immediately revealed to both parties. The coding scheme is simply resending the erased bits, yielding an average rate of \( 1 - \varepsilon \), which is exactly Shannon’s capacity for the BEC. In addition, since all the channel errors are marked as erasures, the probability of decoding error is zero.

In the interactive communication setup for a general BMS channel (other than the BEC), channel errors are not necessarily marked as erasures and perfect feedback is not present. However, the fact that the parties have (a noisy) two-way communication link, enables them to construct a coding scheme in a similar spirit as follows. The parties start by simulating the transcript in a window (or a block) of \( k \) consecutive bits, operating as if the channel is error free. The probability of error in the window can be upper bounded using the union bound by \( k \varepsilon \), and this number is assumed to be small. Next, the parties exchange bits in order to decide if the window is correct, i.e., no errors occurred, which would lead to the simulation of the consecutive window, or incorrect, i.e., some errors occurred, which would lead to retransmission (i.e. re-simulation of the window).

Unfortunately, error detection using less than \( k \) bits of communication has an inherent failure probability. In addition, performing the error detection over a noisy channel can cause further errors, including
a disagreement between the parties regarding the mere presence of the errors. For this purpose, the error
detection is done in a hierarchical and layered fashion. Namely, after \( k \) windows are simulated, error
detection is applied on all of them, including on the outcome of the previous error detections, possibly
initiating their entire retransmission. After \( k^2 \) windows are simulated, error detection is applied on all of
them, and so on. An illustrated example for this concept for \( k = 4 \) is given in Table I.

We are now ready to describe the coding scheme. We note that it can be viewed both as a sequential
algorithm and as a recursive algorithm. For sake of clarity and simplicity of exposition, we chose the
sequential interpretation for the description and the recursive interpretation for the analysis.

A. Building blocks

In the sequel we assume that the order of speakers is alternating, Alice speaking at odd times and Bob
speaking at even times. We denote the input of a the channel by \( X_i \) and its corresponding output by \( Y_i \).
The following notions are used as the building blocks of the scheme:

- The uncoded simulation of the transcript is a sequence of bits, generated by the parties and the
channel, using the transmission functions in \( \pi \) and disregarding the channel errors. Alice’s and
Bob’s uncoded simulation vectors are for odd \( i \) : \( X^A_i \triangleq (X_1, Y_2, \ldots, X_i) \), and \( X^B_i \triangleq (Y_1, X_2, \ldots, Y_i) \)
respectively. For even \( i \) they are \( X^A_i \triangleq (X_1, Y_2, \ldots, Y_i) \), and \( X^B_i \triangleq (Y_1, X_2, \ldots, X_i) \) respectively.

- The cursor variables indicate the time indexes of the transmission functions (i.e. the appropriate
function in \( \pi \)) used by Alice or Bob in the previous transmission. We denote Alice’s and Bob’s
cursors by \( j^A \) and \( j^B \) respectively. We note that \( j^A \) and \( j^B \) are random variables and may not be
identical.

- The rewind bits are the result of the error detection procedure and are calculated at predetermined
points throughout the scheme. They determine whether the simulation of the transcript should proceed
forward, or rewind. We denote \( T = kL \) and separate the rewind bits into layers : \( l = 1, \ldots, L \). At
layer \( l \) there are \( k^{L-l} \) rewind bits, denoted by \( b^A_l(1), \ldots, b^A_l(k^{L-l}) \) for Alice and \( b^B_l(1), \ldots, b^B_l(k^{L-l}) \)
for Bob. The value of Alice’s and Bob’s rewind bits might differ in the general case. The rewind bits
\( b^A_l(m) \) and \( b^B_l(m) \) are calculated after exactly \( mk^l \) bits of uncoded simulation, and are calculated
according to their respective rewind windows. In the sequel we use the term active to denote that a
rewind bit is set to one, and inactive if it is set to zero.

- The rewind window \( w[b^A_l(m)] \) of Alice (resp. \( w[b^B_l(m)] \) of Bob) contains the bits according to
which \( b^A_l(m) \) (resp. \( b^B_l(m) \)) is calculated. It contains the uncoded simulation bits of the respective
party, between times \((m-1)k^l + 1 \) and \( mk^l \). In addition it contains all the rewind bits of levels
\( 1 \leq l < L \) the party has calculated between these times.

We note, that at every point of the simulation, having the uncoded simulation bits and the rewind bits
Calculated so far, both parties can calculate their cursors

\[ i = 0, j^A = j^B = 0, X^A_0 = X^B_0 = \emptyset, \hat{\tau}^A = \hat{\tau}^B = \emptyset, \text{ where } \emptyset \text{ denotes an empty vector. } \]

Iteration:

- Simulate the transcript for \( k \) consecutive times, disregarding the channel errors, as follows. The
parties start by advancing \( i \) and their respective cursors, \( j^A, j^B \) by one. At odd cursors Alice sends
\( X_i = \phi^A_{j^A} (\hat{\tau}^{A-1}_A) \), and at even cursors Bob sends \( X_i = \phi^B_{j^B} (\hat{\tau}^{B-1}_B) \).
At odd cursors Alice updates her uncoded simulation vector by \( X^A_i = (X^A_{i-1}, X_i) \) and her simulation of the transcript by

The coding scheme

Initialization: \( i = 0, j^A = j^B = 0, X^A_0 = X^B_0 = \emptyset, \hat{\tau}^A = \hat{\tau}^B = \emptyset, \text{ where } \emptyset \text{ denotes an empty vector. } \)
\[ \hat{\tau}_A^{j^A} = (\hat{\tau}_A^{j^A-1}, X_i) \] whereas Bob updates \( X_i^B = (X_i^{B-1}, Y_i) \) and \( \hat{\tau}_B^{j^B} = (\hat{\tau}_B^{j^B-1}, Y_i) \). The update for even cursors is done similarly with appropriate replacements. We note that since the block length \( k \) is a power of two (and is therefore even) and since rewinding is done in full blocks, the parties will agree on the parity of the cursors even in the case where their cursors differ. Thus, the parties will always agree which one of them transmits, at every time point.

- For \( l = 1 \) to \( L \), if \( i = mk^l \) for some integer \( m \), then rewind windows \( w \left[ b^A_i(m) \right] \) and \( w \left[ b^B_i(m) \right] \) have ended. Alice computes her rewind bit \( b^A_i(m) \) according to the procedure explained in the sequel. If \( b^A_i(m) = 0 \) she does nothing. If \( b^A_i(m) = 1 \) she rewinds \( j^A \) to the value it had at the beginning of \( w \left[ b^A_i(m) \right] \) and deletes the corresponding values from \( \hat{\tau}_A \). She also sets all the bits of \( w \left[ b^A_i(m) \right] \) in her uncoded simulation vector to zero, so they will not be re-detected as errors in the future. Bob does the same with the appropriate replacements.

### Calculation of the rewind bits

For the first layer, \( l = 1 \), the rewind bits are calculated using the algorithm for error detection using an extended-Hamming code, described in Definition \( \text{D} \). The reason for the choice of this procedure is the fact that in the first layer the difference between \( X^A \) and \( X^B \) is only the channel noise, which is i.i.d. Bernoulli(\( \epsilon \)), and the fact that the extended-Hamming code is a good error detection code for such a noise. In particular, this code is proper [12], which means that the probability of error mis-detection is monotonically increasing for \( 0 < \epsilon < 1/2 \). As the probability of mis-detection for \( \epsilon = \frac{1}{2} \) is equal to that of random hashing with the same number of bits, for \( \epsilon < \frac{1}{2} \) we obtain favorable performance without randomness. The procedure is implemented as follows:

1. Alice calculates the syndrome vector \( s^A \) as explained in Definition \( \text{D} \) according to appropriate rewind window \( w \left[ b^A_i(m) \right] \). She then sends the channel using \( \hat{a} \) repetitions per bit.
2. Bob decodes Alice’s syndrome \( \hat{s}^A \) using a majority vote for every bit. He then calculates his syndrome \( s^B \) according to \( w \left[ b^B_i(m) \right] \) and sets his rewind bit to \( b^B_i(m) = 1 \) \((\hat{s}^A \neq s^B)\).
3. Bob sends Alice \( b^B_i(m) \) using \( \hat{a} \) repetitions per bit. Alice sets \( b^A_i(m) \) according to the respective majority vote.

For all other layers, \( l > 1 \), the procedure is implemented according to the polynomial based randomized error detection scheme from Definition \( \text{E} \). We start by assuming that the parties agree on the prime number \( q_l \) for every layer \( l > 1 \). We also assume for simplicity of exposition, that for every rewind window, the parties commonly and independently draw a test point \( U \) using a common random string. We denote the set comprising all the test points used by the scheme by \( \mathcal{U} \), which contains \(|\mathcal{U}| = O(n)\) elements. In Section \( \text{E} \) we show how the common randomness assumption can be relaxed. The error detection is implemented as follows:

1. Alice uses the appropriate test point \( U \) and the bits of the rewind window \( w \left[ b^A_i(m) \right] \) to calculate \( A(U, w \left[ b^A_i(m) \right]) \in \mathbb{F}_{q_l} \). She then sends the bits representing \( A(U, w \left[ b^A_i(m) \right]) \) to Bob over the channel using \( \hat{a} + 2l \) repetitions per bit.
2. Bob decodes \( \hat{A}(U, w \left[ b^A_i(m) \right]) \) and calculates \( B(U, w \left[ b^B_i(m) \right]) \).
3. Bob calculates his rewind window to \( b^A_i(m) = 1 \) \( \left( \hat{A}(U, w \left[ b^A_i(m) \right]) \neq B(U, w \left[ b^B_i(m) \right]) \right) \) and sends it to Alice using \( \hat{a} \) repetitions.
4. Alice sets \( b^A_i(m) \) according to her respective majority vote.

Let us now bound the number of bits required for this procedure. First, we generously bound the number of bits in a rewind window of layer \( l \), which contains all the uncoded simulation bits and the nested rewind bits of the previous layers, by \( 2k^l \). For layer \( l \), the parties set \( q_l \) to be the first prime number between \( 2k^{2+l} \) and \( 4k^{2+l} \). Therefore, a number in \( \mathbb{F}_{q_l} \) can be represented by no more than \( 2 + (2 + l) \log k \)
bits. All in all the procedure described above required \(3 + (2 + l) \log k\) bits for layer \(l\). For simplicity of calculation, from this point on, we bound this number by

\[
3 + (2 + l) \log k < 3l \log k, \tag{11}
\]

which applies for any \(l \geq 2\) and \(k \geq 4\).

\section{Analysis of the Coding Scheme: A Proof of Theorem 3}

We start by giving the following notation:
- \(j \triangleq \min\{j^A, j^B\}\) is the minimum between Alice’s and Bob’s cursor at any moment
- \(j(T), j^A(T), j^B(T)\) denote the respective values of \(j, j^A, j^B\) at the end of the simulation
- \(\tilde{\tau}_A^{j(T)}\) and \(\tilde{\tau}_B^{j(T)}\) denote the first \(j(T)\) bits of Alice’s and Bob’s simulations of the transcript respectively, at the end of the simulation. We also assume that if \(j^A(T) > n\) or \(j^B(T) > n\) then the parties proceed the protocol by transmitting zeros
- We denote \(b_l(m) \triangleq b_l^A(m) \lor b_l^B(m)\). Namely, \(b_l(m)\) it is defined as the disjunction between Alice’s and Bob’s respective rewind bits

The following two error events will be analyzed

- \(\mathcal{E}_1\) is the event in which \(j(T) < n\)
- \(\mathcal{E}_2\) is the event in which either \(\tilde{\tau}_A^{j(T)} \neq \tau^{j(T)}\) or \(\tilde{\tau}_B^{j(T)} \neq \tau^{j(T)}\)

The simulation error event is included in \(\mathcal{E}_1 \cup \mathcal{E}_2\) and we would like it to vanish with \(n\).

We start by analyzing \(\Pr(\mathcal{E}_1)\) and do it by lower bounding \(j(T)\). We recall that by construction of the scheme, \(b_l^A(m) = 1\) (resp. \(b_l^B(m) = 1\)) will rewind \(j^A\) (resp. \(j^B\)) to the value it had at the beginning of the rewind window. Namely \(j^A\) (resp. \(j^B\)) will be reduced by at most \(k^l\). It is now instrumental to use the definitions of \(j\) and \(b_l(m)\) and observe that if either \(b_l^A(m) = 1\) or \(b_l^B(m) = 1\) (namely, if \(b_l(m) = 1\)) then the minimal among \(j^A\) and \(j^B\) (namely, \(j\)) will be reduced by at most \(k^l\). Recalling that \(T = k^L\) we can now write

\[
j(T) \geq T - \sum_{l=1}^{L} \sum_{m=1}^{k^{L-l}} b_l(m) k^l = T \left(1 - \sum_{l=1}^{L} \overline{b_l}\right), \tag{12}\]

where

\[
\overline{b_l} \triangleq \frac{\sum_{m=1}^{k^{L-l}} b_l(m)}{k^{L-l}} \tag{13}
\]

denotes the average number of active (i.e., non-zero) rewind bits at level \(l\). We note that by construction of the scheme (including its use of randomness), the processes of the error generation and detection are identical for all blocks at level \(l\). For this reason, the probability of having an active rewind bit is also identical for all the blocks at level \(l\). We denote this probability by

\[
P_{b_l} = \Pr(b_l(1) = 1) = \ldots = \Pr(b_l(k^{L-l}) = 1).
\]

Taking the expectation over (12) yields

\[
\mathbb{E}j(T) \geq T \left(1 - \sum_{l=1}^{L} P_{b_l}\right).
\]

In order to proceed with the calculation of \(P_{b_l}\), we define \(P_{el}\) as the probability that either \(b_l^A(m)\) or \(b_l^B(m)\) differ from the error indicator \(1 (w[b_l^A(m)] \neq w[b_l^A(m)])\). This probability does not depend on \(m\) due to the same considerations as above.
Start the simulation: Initialize the cursors: \( j^A = j^B = 0 \)

<table>
<thead>
<tr>
<th>( w[\text{b}_1(1)] )</th>
<th>( \text{b}_1(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 1, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 1, 1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_1(1)] \): No errors, continue. \( j^A = j^B = 4 \)

<table>
<thead>
<tr>
<th>( w[\text{b}_1(1)] )</th>
<th>( \text{b}_1(1) )</th>
<th>( w[\text{b}_1(2)] )</th>
<th>( \text{b}_1(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 1, 1</td>
<td>0</td>
<td>1, 0, 0, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 1, 1</td>
<td>0</td>
<td>1, 0, 0, 1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_1(2)] \): No errors, continue. \( j^A = j^B = 8 \)

<table>
<thead>
<tr>
<th>( w[\text{b}_1(1)] )</th>
<th>( \text{b}_1(1) )</th>
<th>( w[\text{b}_1(2)] )</th>
<th>( \text{b}_1(2) )</th>
<th>( w[\text{b}_1(3)] )</th>
<th>( \text{b}_1(3) )</th>
<th>( w[\text{b}_1(4)] )</th>
<th>( \text{b}_1(4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 1, 1</td>
<td>0</td>
<td>1, 0, 0, 1</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>1</td>
<td>0, 1, 1, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 1, 1</td>
<td>0</td>
<td>1, 0, 0, 1</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>1</td>
<td>0, 1, 1, 1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_1(3)] \): An error occurred and was detected by both parties: \( b^A_1(3) = b^B_1(3) = 1 \)

Both parties zero the rewind window and rewind the cursors to the value it had before the window started: \( j^A = j^B = 8 \)

<table>
<thead>
<tr>
<th>( w[\text{b}_1(4)] )</th>
<th>( \text{b}_1(4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 1, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 1, 1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_1(4)] \): There are no errors so Bob calculates \( b^B_1(3) = 0 \) and continues \( (j^B = 12) \).

However due to an error in communicating \( b^B_1(3) \), Alice decodes \( b^A_1(3) = 1 \), zeros the window and rewinds the cursor \( (j^A = 8) \)

<table>
<thead>
<tr>
<th>( w[\text{b}_2(1)] )</th>
<th>( \text{b}_2(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 1, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 1, 1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_2(1)] \). Calculate \( b_2(1) \).

The errors are detected so \( b^A_2(1) = b^B_2(1) = 1 \), and the cursors are rewound to the beginning of the window: \( j^A = j^B = 0 \).

<table>
<thead>
<tr>
<th>( w[\text{b}_2(1)] )</th>
<th>( \text{b}_2(1) )</th>
<th>( w[\text{b}_3(1)] )</th>
<th>( \text{b}_3(1) )</th>
<th>( w[\text{b}_3(2)] )</th>
<th>( \text{b}_3(2) )</th>
<th>( w[\text{b}_3(3)] )</th>
<th>( \text{b}_3(3) )</th>
<th>( w[\text{b}_3(4)] )</th>
<th>( \text{b}_3(4) )</th>
<th>( w[\text{b}_3(5)] )</th>
<th>( \text{b}_3(5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>0, 0, 0, 0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

End of \( w[\text{b}_3(5)] \) The first four bits of the protocol are re-simulated. No errors. \( j^A = j^B = 4 \).

**TABLE I**

Example for a rewind-if-error coding scheme with \( k = 4 \). Detected error are in **bold**, zeroed bits are in *blue.*
The following lemma bounds $P_{el}$:

**Lemma 5.** For $l = 1$

$$P_{e1} \leq \frac{1}{2k} \left( 1 + 2(k-1)(1-2\varepsilon)^{\frac{2}{k}} + (1-\varepsilon)^k \right) - (1-\varepsilon)^k + (3+\log k)\beta^a,$$

and for $l > 1$

$$P_{el} \leq k^{-l} \left( kP_{e1} + 3\beta^{a+4}k^2 \log k \frac{2-\beta^2k}{(1-\beta^2k)^2} \right).$$ (14)

**Proof.** For the first layer

$$P_{e1} \leq \Pr (NEQ = 0, X^A \neq X^B) + (3+\log k)\beta^a,$$ (15)

where $\Pr (NEQ = 0, X^A \neq X^B)$ is the error mis-detection probability of the extended-Hamming code based error detection scheme of Definition 4 as given in (6). $\beta^a$ is the probability of error in the decoding of a bit sent with $\tilde{a}$ repetitions according to Lemma 8. The multiplication by $(3+\log k)$ accounts for the union bound over the number of bits used for the error detection: $2+\log k$ bits sent from Alice to Bob $(1+\log k)$ required for the description of the syndrome according to Lemma 2 and an additional bit reserved for avoiding the random tie breaking as described in Subsection IX-C and a single bit fed back from Bob to Alice.

The key idea in the analysis of the scheme for $l > 1$ is regarding the calculation of the rewind bits as a layered recursive process. Namely, we observe that by construction, a rewind window at level $l$ comprises $k$ rewind windows of level $l-1$. In addition, the polynomial based randomized error detection of Definition 3 uses independent test points for every layer and hence is independent between layers.

Having this notion we can write the following recursion formula:

$$P_{el} \leq k^{-2}kP_{el-1} + (2 + (2 + l)\log k)\beta^{a+2l},$$ (16)

where $kP_{el-1}$ is the union bound over the error events of the previous level. The multiplication by $k^{-2}$ accounts for the probability of error mis-detection according to Lemma 3 with the setting $\gamma = k^{-2}$ and $\ell$ as the number of bits in the appropriate rewind window $w \ [b^A(t)]$ (or $w \ [b^B(t)]$). As described above, for the error detection, Alice should send Bob a number in $\mathbb{F}_q$ and Bob should reply with a single bit (we assume that the set of test points $\mathcal{U}$ is jointly drawn by the parties using common randomness). We recall that the number of bits required for the error detection scheme of Definition 3 is generously bounded by $3l \log k$ due to (11). All in all, we can rewrite (16) as

$$P_{el} \leq k^{-1}kP_{el-1} + 3\beta^a(\log k)\beta^{2l},$$ (17)

Solving the recursion of (12) with the initial condition in (15) we can bound $P_{el}$ as follows:

$$P_{el} \leq k^{-1}kP_{el-1} + 3\beta^a(\log k)k^{-l} \sum_{j=2}^{l} j\beta^2k^{j-l} \leq k^{-1}kP_{el-1} + 3\beta^a(\log k)k^{-l} \sum_{j=2}^{\infty} j(\beta^2k)^j$$ (18)

$$= k^{-1}kP_{el-1} + 3\beta^a(\log k)k^{-l}(\beta^2k)^2 \frac{2-\beta^2k}{(1-\beta^2k)^2} = k^{-1} \left( kP_{el-1} + 3\beta^{a+4}k^2 \log k \frac{2-\beta^2k}{(1-\beta^2k)^2} \right).$$
We note that the assumption in Theorem 3 that $\varepsilon < 1/(8k)$ assures that $\beta^2 k < 1$ assuring that the infinite sum in (18) converges.

We are now ready to bound $P_b$. We recall that it is defined as the probability that either $b^A_l(m) = 1$ or $b^B_l(m) = 1$, and is independent of $m$ due to the symmetry of the scheme. For $l = 1$ we use the union bound over the probability of an erroneous bit and a communication error:

$$P_b \leq k\varepsilon + (3 + \log k)\beta^a \triangleq P_b^1.$$ 

Similarly, for $l > 1$ we take the union bound over the probability of error $P_e l_1$, in one of the $k$ blocks in the layer $l - 1$ and a communication error:

$$P_b \leq kP_e l_1 + 3\beta^a (\log k)l\beta^{2l}$$

$$\leq k^{2-l} \left( kP_e + 3\beta^a + k^2 \log k \frac{2 - \beta^2 k}{(1 - \beta^2 k)^2} \right) + 3\beta^a (\log k)l\beta^{2l}$$

$$\triangleq P_b^l.$$ (19)

Let us now bound the average rewind by

$$E_j(T) \geq T \left( 1 - \sum_{l=1}^\infty P_b l \right) = T\zeta.$$ (20)

where

$$\zeta \triangleq 1 - \sum_{l=1}^\infty P_b l$$

$$= 1 - k\varepsilon - (3 + \log k)\beta^a - \frac{k^2}{k-1} \left( P_e + 3\beta^a + k^2 \log k \frac{2 - \beta^2 k}{(1 - \beta^2 k)^2} \right) - 3\beta^a + 4k^2 \log k \frac{2 - \beta^2 k}{(1 - \beta^2 k)^2}.$$ 

Setting

$$T = \frac{n}{1 - \sum_{l=1}^\infty P_b l - \xi} = \frac{n}{\zeta - \xi}$$ (21)

for some $0 < \xi < \zeta$ will therefore ensure that $E_j(T) \geq n$. The following lemma assures that $Pr(\mathcal{E}_1)$ also vanishes in $n$:

**Lemma 6.** For any $\xi > 0$ and $T$ that satisfies (21):

$$\lim_{n \to \infty} Pr(\mathcal{E}_1) = \lim_{n \to \infty} Pr(j(T) < n) = 0.$$ 

The proof is in Appendix B. It is based on the fact that due to (20) and (21) we have $E_j(T) \geq (1 + \eta)n$ for some $\eta > 0$ and using standard concentration techniques. We note that the proof assumes the number of test points in $\mathcal{U}$ is $|\mathcal{U}| = O(\sqrt{n})$, whereas so far we assumed that every use of the error detection procedure of Definition 3 uses a different test point (i.e. $|\mathcal{U}| = O(n)$). Since $|\mathcal{U}| = O(\sqrt{n})$ is restrictive, Lemma 6 also holds for the current description of the scheme. The motivation for reducing $|\mathcal{U}|$ is changing the common randomness to private randomness, which is extracted from the channel, and is elaborated in Section IX.

The following lemma assures $Pr(\mathcal{E}_2)$ vanishes in $n$:
Lemma 7. For any \( \xi > 0 \) and \( T \) that satisfies \((21)\)
\[
\lim_{n \to \infty} \Pr(\mathcal{E}_2) = 0.
\]

Proof. We remind the reader that \( P_{el} \) is defined as the probability that either \( b_i^A(m) \) or \( b_i^B(m) \) differ from the error indicator \( w[\{b_i^A(m)\}] \neq w[\{b_i^A(m)\}] \). Namely, it is the probability of an undetected error, or a falsely detected error, in the simulation of a block in layer \( l \) at least at one party. Since \( L \) is the final layer, and due to the recursive structure of the error detection, \( P_{eL} \) therefore upper bounds the respective probability at the end of the coding scheme. The error event related to \( P_{eL} \) includes \( \mathcal{E}_2 \) and therefore \( \Pr(\mathcal{E}_2) \leq P_{eL} \). Rewriting \((12)\) and setting \( l = \log_k T = \log_k(n/(\zeta - \xi)) \) we obtain:
\[
\Pr(\mathcal{E}_2) \leq \frac{\zeta - \xi}{n} \left( kP_{e1} + 3\beta^{a+4}k^2 \log k \frac{2 - \beta^2k}{(1 - \beta^2k)^2} \right).
\]
Therefore \( \lim_{n \to \infty} \Pr(\mathcal{E}_2) = 0. \)

Let us now bound \( N \), the total number of channel uses consumed by the scheme:
\[
N \leq T + \tilde{a}(3 + \log k)k^{L-1} + 3 \log k \sum_{l=2}^{\infty} l(a + 2l)k^{L-l}
\leq T \left( 1 + \frac{\tilde{a}(3 + \log k)}{k} + 3 \log k \left[ \frac{a(2k-1)}{(k-1)^2} + \frac{4k}{(k-1)^3} + \frac{4k-2}{k(k-1)^2} \right] \right), \tag{22}
\]
where \( \tilde{a}(3 + \log k)k^{L-1} \) is the number of channel uses required for the error detection at the first layer, and \( 3 \log k \sum_{l=2}^{\infty} l(a + 2l)k^{L-l} \) is the number of channel uses required for the error detection in all other layers. Using \((21)\) and \((22)\) we can bound the total rate of the scheme:
\[
R_{BSC}(\varepsilon, k) \geq \frac{1 - k\varepsilon - (3 + \log k)\beta^{a+4}k^2}{k-1} \left( P_{e1} + 3\beta^{a+4}k \log k \frac{2 - \beta^2k}{(1 - \beta^2k)^2} \right) - 3\beta^{a+4}k^2 \log k \frac{2 - \beta^2k}{(1 - \beta^2k)^2} - \xi.
\]
Since this holds for any \( \xi > 0 \), we can take the limit \( \xi \to 0 \) and conclude the proof of Theorem \( 3. \)

VIII. Generalization to Binary Memoryless Symmetric Channels

In Definition \( 4. \) we defined a binary memoryless symmetric (BMS) channel as a collection of BSC's with various crossover probabilities. The simplest example for a BMS channel is the BSC(\( \varepsilon \)) for which \( T = \varepsilon \) with probability one. A special case with a continuous output alphabet is the binary additive white Gaussian noise (BiAWGN) channel, \( Y = X + Z \) where \( X \in \{-1, +1\} \) and \( Z \sim \mathcal{N}(0, \sigma^2) \) is statistically independent of \( X \), for which \( T \) is a continuous random variable on \([0, 1/2]\). The binary erasure channel with erasure probability \( \epsilon \), BEC(\( \epsilon \)), can be cast as a BMS channel taking \( T = \frac{1}{2} \) with probability \( \epsilon \) and \( T = 0 \) with probability \( 1 - \epsilon \). It is in place to note, however, that in an actual BEC, a Bernoulli(1/2) bit is not produced when \( T = 1/2 \). This delicate point is discussed in Subsection \( IX-C. \)

We now extend the notion of repetition coding of Lemma \( 4. \) to BMS channels.

Definition 4. \( \rho \)-repetition channel Let \( P^\rho_{\tilde{Y}|X} \) be the \( \rho \)-repetition channel corresponding to a BMS(\( P_Y|X \)) channel, obtained by transmitting \( \rho \) repetitions of the bit \( \tilde{X} \) through a BMS(\( P_Y|X \)) channel and taking
\[
\tilde{Y} = \arg\max_{x \in \{0, 1\}} \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|x),
\]
where ties are broken by drawing a Bernoulli(1/2) random variable.

We note that like in the BSC case, we randomly break the ties in order to facilitate the analysis and later explain in Subsection IX-C how this random procedure can be circumvented. The following lemma bounds the decoding error of the ρ-repetition channel.

**Lemma 8.** For any BMS(\(P_{Y|X}\)) channel with Shannon capacity \(C_{\text{Sh}}(P_{Y|X}) = C\) the corresponding ρ-repetition channel \(P_{Y|X}^{(\rho)}\) is a BSC(\(\delta\)) with \(\delta \leq \beta^\rho\), where \(\beta = 2\sqrt{h^{-1}(1 - C) \cdot (1 - h^{-1}(1 - C))}\) is the Bhattacharyya parameter of a BSC(\(\varepsilon\)) with capacity \(C\).

**Proof.** We start by rewriting the maximum-likelihood decision rule from Definition 4:

\[
\Lambda \triangleq \ln \left[ \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|0) \right]_{\tilde{Y}=0}^{\tilde{Y}=1} 0
\]

Using the sufficient statistic \(g(Y) = (X \oplus Z_T, T)\) from Definition 1, it is easy to show that the log-likelihood function \(\Lambda\) can be written as

\[
\Lambda = (-1)^X \sum_{i=1}^{\rho} (1 - 2Z_{Ti}) \ln \frac{1 - T_i}{T_i}.
\] (23)

The (symmetric) decision error probability can now be upper bounded by

\[
\delta = \Pr(\tilde{Y} \neq \tilde{X}) = \Pr(\tilde{Y} \neq \tilde{X} | \tilde{X} = 0) \leq \Pr\left( (-1)^X \sum_{i=1}^{\rho} (1 - 2Z_{Ti}) \ln \frac{1 - T_i}{T_i} \leq 0 \right) \tag{24}
\]

\[
= \Pr\left( \sum_{i=1}^{\rho} (1 - 2Z_{Ti}) \ln \frac{1 - T_i}{T_i} \leq 0 \right). \tag{25}
\]

We note that the inequality in (24) implies that the event of a tie (i.e., \(\Lambda = 0\)) is regarded as an error in probability one, where in fact, due to the random tie breaking, it is an error with probability half. We now recall the Chernoff bound for a sum of i.i.d. random variables \(A_1, ..., A_\rho:\)

\[
\Pr\left( \sum_{i=1}^{\rho} A_i \leq a \right) \leq e^{sa} \left[ Ee^{-sA_i} \right]^\rho.
\]

for any \(s > 0\). Applying this bound to (25) with \(A_i = (1 - 2Z_{Ti}) \ln \frac{1 - T_i}{T_i}, a = 0\) and \(s = 1/2\) yields

\[
\delta \leq \beta^\rho \tag{26}
\]
where $\beta$ is defined as the Bhattacharyya parameter of the channel $P_{Y|X}$, which is equal to:

$$
\beta = \mathbb{E}_{T,Z_T} \left( \left( \sqrt{\frac{T}{1-T}} \right)^{1-2Z_T} \right) \\
= \mathbb{E}_T \left( \mathbb{E}_{Z_T|T} \left( \left( \sqrt{\frac{T}{1-T}} \right)^{1-2Z_T} \mid T \right) \right) \\
= \mathbb{E} \left( 2\sqrt{T(1-T)} \right).
$$

It was shown by Guillén i Fàbregas et al. [8] that among all BMS channels $P_{Y|X}$ with capacity $C$, the Bhattacharyya parameter is maximized by a BSC. Their proof is based on the fact that the function $x \mapsto \sqrt{h^{-1}(x) \cdot (1 - h^{-1}(x))}$ is concave, and therefore:

$$
\beta = \mathbb{E}[2\sqrt{T(1-T)}] \\
= 2\mathbb{E} \left[ \sqrt{h^{-1}(h(T)) \cdot (1 - h^{-1}(h(T)))} \right] \\
\leq 2\sqrt{h^{-1}(\mathbb{E}[h(T)]) \cdot (1 - h^{-1}(\mathbb{E}[h(T)]))} \\
= 2\sqrt{h^{-1}(1-C) \cdot (1 - h^{-1}(1-C))} \\
= 2\sqrt{\varepsilon \cdot (1-\varepsilon)} \\
= \beta
$$

where in (27) we used Jensen’s inequality, in (28) we used the fact that capacity of a BMS channel is $C = 1 - \mathbb{E}[h(T)]$, and in (29) we used the capacity of the BSC($\varepsilon$) $C = 1 - h(\varepsilon)$. Combining (26) and (29) concludes the proof of the lemma.

We are now ready to prove Theorem 1, which is a generalization of Theorem 2 to BMS channels.

**Proof of Theorem 1.** We follow the same lines as in the in proof of Lemma 4 and start by converting the BMS($P_{Y|X}$) channel to a BSC($\delta'$) with $0 < \delta' \leq \delta$. According to Lemma 8 this can be done using

$$
\rho(P_{Y|X}, \delta) \triangleq \frac{\log \frac{1}{\beta}}{\log \frac{1}{\beta}} + 1.
$$

repetitions where $\beta = 2\sqrt{\varepsilon(1-\varepsilon)}$ is the Bhattacharyya parameter of a BSC($\varepsilon$) with capacity $C_{Sh}(\varepsilon) = C_{Sh}(P_{Y|X})$. We then apply an interactive coding scheme for the BSC($\delta$) with rate $R(\delta)$. After normalizing by $C_1(P_{Y|X})$ the following bound it obtained:

$$
\frac{C_1(P_{Y|X})}{C_{Sh}(P_{Y|X})} \geq \frac{R(\delta)}{\rho(P_{Y|X}, \delta)C_{Sh}(P_{Y|X})}. \\
\text{(30)}
$$

Bounding the denominator of the right hand term in (30):

$$
\rho(P_{Y|X}, \delta)C_{Sh}(P_{Y|X}) = \left( \frac{\log \frac{1}{\beta}}{\log \frac{1}{\beta}} + 1 \right) C_{Sh}(P_{Y|X}) \\
= \left( \frac{\log \frac{1}{\beta}}{\log \frac{1}{\beta}} + 1 \right) C_{Sh}(\varepsilon),
$$

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which is exactly 10. The rest of the proof is as in Lemma 4, and using the same coding scheme to obtain the same numeric value in the lower bound as in Theorem 2.

For completeness, we now show that not only \( \frac{C_I(P_{Y|X})}{C_{Sh}(P_{Y|X})} \geq 0.0302 \) for any BMS channel, but also the ratio \( \frac{C_I(P_{Y|X})}{C_{Sh}(P_{Y|X})} \) tends to one as the BMS channel becomes cleaner, similarly to the BSC case.

**Corollary 2.** For any sequence in \( C \) of BMS channels \( P^C_{Y|X} \) with \( C_{Sh}(P^C_{Y|X}) = C \), we have

\[
\lim_{C \to 1} \frac{C_I(P_{Y|X})}{C} = 1.
\]

**Proof.** We start by proving that without repetitions a BMS\((P_{Y|X})\) channel can be reduced to BSC\((\varepsilon)\) with \( \varepsilon \leq \frac{1-C_{Sh}(P_{Y|X})}{2} \). As in [21], the proof is by noting that \( h(t) \geq 2t \) for any \( t \in [0,1/2] \) and therefore:

\[
C_{Sh}(P_{Y|X}) = 1 - \mathbb{E}h(T) \\
\leq 1 - \mathbb{E}2T \\
= 1 - 2\varepsilon.
\]

The corollary now follows by taking the lower bound for \( R_{BSC}(\varepsilon, k) \) in Corollary 1 as a lower bound to \( C_I(P_{Y|X}) \).

**IX. A DETERMINISTIC CODING SCHEME**

The coding scheme described throughout this paper uses randomness for two purposes: the randomized polynomial based error detection procedure described in Definition 3, and the random tie breaking in the repetition decoding described in Lemma 1 and Lemma 8. In this section we show how the requirements for randomness can be relaxed using a few simple adaptations of the coding scheme.

**A. On the randomness requirements of the error detection scheme in Definition 3**

We start by recalling that the scheme from Definition 3 requires a random generation of a test point \( U \) taken from a finite field. We note that original scheme from [11, p. 30] requires only private randomness. Namely, the test point \( U \) should be drawn by Alice party and conveyed to Bob. However, so far we assumed that the all test points used by the scheme (denoted by \( U \)) are jointly drawn by both parties using a shared random string (i.e., public randomness). This choice was made in order to save the communication overhead of conveying the test points from one party to the other, which is prone to reduce the overall rate of the interactive communication scheme.

The first step in modifying the communication scheme to private randomness is showing the number of random test points can be reduced, without affecting the overall rate. We start showing that \( |U| \), the number of random test points required for all the error detections in the interactive coding scheme can be reduced to \( o(n) \). This way, if only private randomness is used, \( U \) can be reliably conveyed from one party to the other without affecting the total rate. In Subsection IX-B we show how \( U \) can be generated using randomness extracted from the channel, removing the requirement for private randomness.

We start by noting that by construction of error detection scheme, using independently drawn test points for its different actuations, will make their corresponding error mis-detection events statistically independent. It is now in place to discuss the amount of statistical independence required by the coding scheme. In [18] we assumed that the probability of error mis-detection is independent between layers. That might imply that using \( |U| = L \) is satisfactory. In fact, if one is concerned only with the average rate of the coding scheme, using only \( |U| = L \) will lead to the same average rate of Theorem 3.
However, we recall that we defined rate not in the average sense, but rather, we required the reconstruction of the transcript with high probability after a predetermined simulation length. To illustrate this delicate difference, consider the example of the one-way BEC with feedback. In this example, all the erased bits are retransmitted. So, using the channel $n$ times will result in $n(1 - \epsilon)$ bits decoded with zero error, where $\epsilon$ is the erasure probability. This means that the average rate is $1 - \epsilon$, which is exactly the Shannon capacity of the BEC($\epsilon$). However, it is interesting to note that since the erasures are drawn i.i.d., for $n \to \infty$ the rate will concentrate around its average and the probability of decoding less than $n(1 - \epsilon - \xi)$ bits will vanish in $n$ for any $\xi > 0$. This means, that this simple scheme also achieves Shannon’s capacity in a stricter deterministic sense - namely, for $n \to \infty$ a number of information bits respective to Shannon’s capacity could be reliably transmitted with a vanishing error probability using a fixed number of channel uses.

For our scheme, the convergence to the average rate is stated in Lemma 3. The concept of the proof appearing in Appendix B is similar to that of the BEC with feedback. We regard the rewind bits as the counterparts of the erasures in the BEC and show that actual number of rewind bits in every layer, concentrates around its average. A delicate issue in the analysis is the independence of the rewind bits in our scheme. In the first layer, the rewind bits are calculated according to Definition 2. This is a deterministic scheme that is based only on the vectors of channel errors, which are i.i.d between different blocks. Therefore, the rewind bits are indeed i.i.d. For higher layers, the scheme in Definition 3 is used. As explained in the proof of Lemma 3, the rewind bit is calculating according to

$$1 \left( \sum_{i=1}^{\ell} (X_i^A - X_i^B)U^{i-1}(\mod q) \neq 0 \right).$$

While it is tempting to assume that $X_i^A - X_i^B$ is exactly the vector of i.i.d channel errors, we note that the “-” operation is done over $\mathbb{F}_q$ and not over $\mathbb{F}_2$. This means, that the event of error mis-detection depends not only on the channel error vector, but also on the vectors related to the transcript: $X_i^A$, $X_i^B$. Since the transcript might be dependent between consecutive blocks, the corresponding rewind bits might also be statistically dependent, if the same value of $U$ is used for both blocks.

One way of breaking this dependence is drawing independent $U$ for every error detection in every layer. As stated before, if common randomness is used, this procedure is feasible, but when using only private randomness it might cause a decrease of the total rate. We recall that in every layer $1 < l \leq L$, there are $k^{L-l}$ blocks for which error detection is applied using Definition 3. In our modification of the coding scheme for private randomness we assume that only $k^{(L-l)/2}$ independent test points are used, such that the test point is changed every $k^{(L-l)/2}$ blocks. In Appendix 3 we prove that this reduced number of independent test points still assures a slower, yet fast enough, concentration.

Let us now bound the total number of bits required for the description of $U$ denoted by $n_U$. We recall that the number of bits required for the error detection at layer $1 < l \leq L$ is bounded by $3l\log k$ by (11). So, the overall number of bits can be upper bounded by

$$n_U \leq \sum_{l=2}^{L} 3l(\log k)k^{(L-l)/2} \leq 3k^{L/2+1}\log k \sum_{l=2}^{\infty} lk^{-l/2} = O(k^{L/2}\log k) = O(\sqrt{n})$$

These bits can be conveyed from Alice to Bob before the beginning of the simulation using a block code with some constant positive rate $R_U$ below Shannon’s capacity, requiring $\frac{n_U}{R_U} = O(\sqrt{n})$ channel uses. However, an error in the decoding of $U$ might occur, which might cause a failure in the simulation of the entire transcript. We denote this error event by $E_3$ and add it to the previously defined error events.
\(\mathcal{E}_1\) and \(\mathcal{E}_3\). The probability of \(\mathcal{E}_3\) can be upper bounded by an error exponent yielding:
\[
\Pr(\mathcal{E}_3) \leq e^{-O(\sqrt{n})}
\]
so clearly \(\lim_{n \to \infty} \Pr(\mathcal{E}_3) = 0\) making this error event negligible. We should also add \(\frac{n_U}{R}\) to the total number of channel uses of the scheme in (22). But since \(\frac{n_U}{R_U} = O(\sqrt{n})\), \(N\) would change only by \(O(\sqrt{n})\), which would not affect the asymptotic value of the rate from Theorem 3.

**B. Extracting randomness from the channel**

In the previous subsection we showed that the error detection procedure of Definition 3 can be implemented using private randomness requiring \(n_U \leq O(\sqrt{n})\) random bits for the entire coding scheme, which were assumed to be drawn by Alice. Our coding scheme can however, be made explicit by extracting the random bits from the channel. While a randomness extraction procedure with optimal efficiency was presented by Elias in [14], we use von-Neumann’s suboptimal scheme [13] due to its simplicity of analysis and the vanishing effect of its suboptimality on the total rate.

**Lemma 9.** The coding scheme can be made explicit by extracting the randomness from the channel with an overhead of
\[
n_R = O(\sqrt{n})
\]
channel uses and an additional error probability
\[
\Pr(\mathcal{E}_4) \leq e^{-O(\sqrt{n})}.
\]

*Proof.* Bob sends Alice \(n_R\) zeros and Alice receives a noise vector \(Z_1, ..., Z_{n_R}\) whose elements are i.i.d Bernoulli(\(\varepsilon\)). Alice then divides the noise elements into pairs. For the pairs 00 and 11, Alice does nothing. For the pairs 01 or 10 Alice extracts a single random bit valued 0 or 1 respectively. Clearly if a bit was extracted, it is 0 or 1 with equal probability. We now define \(W_i\) as a Bernoulli r.v. that is set to one if a random bit was extracted:
\[
W_i = 1 \text{ if } (Z_{2i-1}Z_{2i} = 01 \lor Z_{2i-1} Z_{2i} = 10),
\]
such that \(\Pr(W_i = 1) = 2\varepsilon(1 - \varepsilon)\). Therefore, the (random) number of extracted bits is
\[
N_R = \sum_{i=1}^{n_R/2} W_i,
\]
and the probability of failure in the random bit extraction is
\[
\Pr(\mathcal{E}_4) = \Pr(N_R < n_U).
\]
We now set
\[
n_R = \frac{n_U}{\varepsilon(1 - \varepsilon)(1 - \delta)} = O(\sqrt{n})
\]
for some fixed \(0 < \delta < 1\). Using the multiplicative form of Chernoff’s bound
\[
\Pr(\mathcal{E}_4) = \Pr\left(\sum_{i=1}^{n_R/2} W_i < (1 - \delta)\mathbb{E}\sum_{i=1}^{n_R/2} W_i\right) \leq e^{-\frac{\delta^2 n_R}{2(1 - \delta)}} = e^{-O(\sqrt{n})}
\]
\(\square\)
whose transition matrix $P$ binary channel with symmetric error and erasure, we proceed it is instrumental to define randomness, and with an improved efficiency.

error detection procedure, the rewind-if-error for the BSC could potentially be used, without requiring an error scheme. However, since the erasure is naturally detected by its receiver without requiring an of the error event in the BSC. A BSC could be applied. However, we note that the erasure event in the BEC, $Y = 1$, will reduce the BEC(δ) to a BSC(δ/2) and the coding scheme designated for a BSC could be applied. However, we note that the erasure event in the BEC(δ) has the same probability of the error event in the BSC(δ), which is to be detected in the error detection phase of the rewind-if-error scheme. However, since the erasure is naturally detected by its receiver without requiring an error detection procedure, the rewind-if-error for the BSC could potentially be used, without requiring randomness, and with an improved efficiency.

We can now extend the notion of treating ties as erasures to the general case of a BMS channel. Before we proceed it is instrumental to define binary channel with symmetric error and erasure, BSEC(δ − ε, ε), whose transition matrix $P_{Y|X}$ appears in Table II. It is clear from the definition that δ ∈ [0, 1/2] and ε ∈ [0, δ], where ε = 0 for a BSC(δ) and ε = δ for a BEC(ε). In addition, it is easy to see that for any $\epsilon \in [0, \delta]$, the capacity of the BSEC(δ − ε, ε) is

$$C_{\text{BSEC}}(\delta, \epsilon) = (1 - \epsilon) \left(1 - h\left(\frac{1 - \delta}{1 - \epsilon}\right)\right),$$

which can be proved by analysis to be strictly larger than $C_{\text{Sh}}(\delta)$ for every $0 < \epsilon \leq \delta$.

We now give a non-random version of Definition II and Lemma VIII, in which ties are marked as erasures:

**Definition 5.** [ρ-repetition channel with erasures] Let $P_{Y|X}^{(\rho, E)}$ be the ρ-repetition channel with erasure, corresponding to a BMS($P_{Y|X})$ channel, obtained by transmitting ρ repetitions of the bit $\tilde{X}$ through BMS($P_{Y|X}$) channel and taking

$$\tilde{Y} = \begin{cases} 0 & \text{if } \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|0) > \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|1) \\ 1 & \text{if } \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|0) < \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|1) \\ E & \text{if } \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|0) = \prod_{i=1}^{\rho} P_{Y_i|X}(Y_i|1) \end{cases}$$

**Lemma 10.** For any BMS($P_{Y|X}$) channel with Shannon capacity $C_{\text{Sh}}(P_{Y|X}) = C$ the corresponding ρ-repetition with erasure channel $P_{Y|X}^{(\rho, E)}$ is a BSEC(δ − ε, ε) with $\epsilon \in [0, \delta]$ and $\delta \leq \beta^\rho$ where $\beta$ is as in Lemma VIII.

**Proof.** The proof follows the same lines as the proof of Lemma VIII by making two observations. The first is by noting that in Definition II it was implied that an erasure event in a BMS channel corresponds to

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<tr>
<th>X</th>
<th>0</th>
<th>E</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 − δ</td>
<td>ε</td>
<td>δ − ε</td>
</tr>
<tr>
<td>1</td>
<td>δ − ε</td>
<td>ε</td>
<td>1 − δ</td>
</tr>
</tbody>
</table>

**TABLE II**

THE TRANSITION MATRIX $P_{Y|X}$ OF A BSEC(δ − ε, ε)
the statistic \( g(Y) = (T, X \oplus Z_T) \) with \( T = 1/2 \) and \( X \oplus Z_T \), which is a Bernoulli(1/2) random bit. In Definition \( \text{Definition } 8 \) as well as in the standard BEC definition, such a bit is not produced. However, we note that in the log-likelihood ratio function used for the decision (23), the value of the random bit is not used. The second observation is by noting that in Lemma \( \text{Lemma } 8 \), ties were pessimistically regarded as errors with probability one, where in fact, the random tie breaking reduces their respective error probability to half. Therefore, marking ties as erasures, the aggregate probability of erasure and error is \( \delta \) and the induced channel is a BSEC(\( \delta - \epsilon, \epsilon \)) with \( \delta \) as in Lemma \( \text{Lemma } 8 \) and \( \epsilon \in [0, \delta] \). □

We are now ready to present the rewind-if-error coding scheme, without tie breaking. We note that ties can appear in two contexts: i) If the original BMS channel had an erasure event (i.e., the probability of \( T = 1/2 \) is strictly positive). ii) If the BMS channel was reduced to BSC using Lemma \( \text{Lemma } 8 \) and ties occurred in the decoding. We note that ties cannot occur in the repetition coding used for the transmission of the error detection bits in the BSC scheme, since the number of repetitions is always odd.

For for contexts the rewind-if-error scheme can be modified as follows: when a party receives an erasure, it uses the zero value in order to calculate its next bit of the transcript. Then, at the end of the corresponding rewind window, the standard error detection procedure is bypassed and an error is announced. If the erasure was detected by Bob, he simply sets the rewind bit to one and sends it to Alice. If it was detected by Alice, she signals a designated symbol to Bob, indicating the erasure. We note that in the first layer an additional bit was reserved for this purpose. In higher layers, the bound in (11) assures that the extra symbol could be signaled without requiring additional bits.

For the sake of completeness, the issue of erasures should also be discussed in the context of randomness extraction in Subsection \( \text{Subsection } \text{IX-B} \). Here, we note that if the channel used for randomness extraction can be reduced to a BSEC(\( \delta - \epsilon, \epsilon \)), with \( \epsilon < \delta \), Lemma \( \text{Lemma } 9 \) could still be used, changing \( n_R \) only by a constant factor and leaving it in an order of magnitude of \( O(\sqrt{n}) \). In the extreme case \( \epsilon = \delta \) (a pure BEC), Lemma \( \text{Lemma } 9 \) could not be used. However, in this case all the errors in the scheme in all layers (including the errors of the repetition used for the error detection bits) are marked as erasure. Therefore, the random error detection procedure of Lemma \( \text{Lemma } 9 \) need not be used, and random bits need not be extracted from the channel.

X. CONCLUDING REMARKS

In this paper we revisited the problem of interactive communication over noisy channels originally introduced by Schulman \( \text{[3]} \), and studied the problem from an information- and communication-theoretic perspective. We started by defining the interactive channel capacity with respect to a protocol and not with respect to a distributed computing problem. As a consequence, our definitions do not use the notion of communication complexity. We then presented a structured and deterministic rewind-if-error coding scheme, and used it to calculate a lower bound for the ratio between the Shannon capacity and the interactive capacity of every BMS channel. To the best of our knowledge, this is the first time that a numerical value is attached to this ratio.

We note that the current value of the lower bound can likely be further improved using different coding schemes. A nontrivial upper bound on the ratio between the Shannon capacity and the interactive capacity for a fixed channel (i.e., not in the limit of a very clean channel) remains an intriguing open question even in the simplest binary symmetric case.
APPENDIX A

PROOF OF COROLLARY 1

We begin by writing (8) as

\[ R_{BSC}(\varepsilon, k) = \frac{1 - A(\varepsilon, k)}{1 + B(\varepsilon, k)} \]

where

\[ A(\varepsilon, k) \triangleq k\varepsilon + (2 + \log k)\beta^a + \frac{k^2}{k-1} \left( P_{e1} + 3\beta^{a+4}k\log k \frac{2 - \beta^2k}{(1 - \beta^2k)^2} \right) + 3\beta^{a+4}k^2 \log k \frac{2 - \beta^2k}{(1 - \beta^2k)^2} + \xi \]

and

\[ B(\varepsilon, k) \triangleq \frac{\tilde{a}(2 + \log k)}{k} + 3\log k \left[ \frac{a(2k - 1)}{(k - 1)^2} + \frac{4k}{(k - 1)^3} + \frac{4k - 2}{k(k - 1)^2} \right] + o(1). \]

Using the inequality \(1/(1 + x) < 1 - x\) for \(x > 0\) and the fact that \(A(\varepsilon, k) \geq 0, B(\varepsilon, k) \geq 0\) gives:

\[ R_{BSC}(\varepsilon, k) \geq 1 - A(\varepsilon, k) - B(\varepsilon, k) + A(\varepsilon, k)B(\varepsilon, k) \]

\[ \geq 1 - A(\varepsilon, k) - B(\varepsilon, k). \]

We use the definitions \(\beta = 2\sqrt{\varepsilon(1 - \varepsilon)}, a = 3\) and \(\tilde{a} = 5\) and assume from this point on that \(k \to \infty\) and \(\varepsilon = o(1/k)\). Neglecting all high order terms we obtain:

\[ B(\varepsilon, k) = O \left( \frac{\log k}{k} \right) \]

and

\[ A(\varepsilon, k) = k\varepsilon + O(k)P_{e1} + \xi + o(1). \]

We now recall (9)

\[ P_{e1} \leq \frac{1}{2k} \left( 1 + 2(k - 1)(1 - 2\varepsilon)^2 + (1 - \varepsilon)^k \right) - (1 - \varepsilon)^k + (2 + \log k)\beta^a = O(k\varepsilon^2). \]

and set

\[ \xi = k^{-2}, \]

which assures that Lemma 1 holds (see (11)) obtaining

\[ R_{BSC}(\varepsilon, k) \geq 1 - \left( k\varepsilon + O(k^2\varepsilon^2) + k^{-2} + o(1) + O \left( \frac{\log k}{k} \right) \right). \]

Finally, setting \(\varepsilon = \frac{\log k}{k^2}\) as in (11) gives

\[ R_{BSC}(\varepsilon, k) \geq 1 - O \left( \frac{\log k}{k} \right) = 1 - O \left( \sqrt{-\log \varepsilon} \right) = 1 - O \left( \sqrt{h(\varepsilon)} \right). \]

APPENDIX B

PROOF OF LEMMA 1

We would like to prove that

\[ \lim_{n \to \infty} \Pr(E_1) = \lim_{n \to \infty} \Pr \left( j^A(T) < n \right) = 0. \]
We start by recalling (12)

\[ j^A(T) \geq T \left( 1 - \sum_{l=1}^{L} b_l \right) \]

The probability of the complementary event is:

\[ \Pr \left( j^A(T) \geq n \right) \geq \Pr \left( 1 - \sum_{l=1}^{L} b_l \geq \frac{n}{T} \right). \] (31)

By (21) we have

\[ \frac{n}{T} = 1 - \sum_{l=1}^{\infty} P_{b_l} - \xi \leq 1 - \sum_{l=1}^{L} P_{b_l} - \xi, \]

so we can further bound (31) by

\[ \Pr \left( j^A(T) \geq n \right) \geq \Pr \left( 1 - \sum_{l=1}^{L} b_l \geq 1 - \sum_{l=1}^{L} P_{b_l} - \xi \right) = 1 - \Pr \left( \sum_{l=1}^{L} b_l > \sum_{l=1}^{L} P_{b_l} + \xi \right). \]

Therefore \( \Pr(\xi_1) \leq \Pr \left( \sum_{l=1}^{L} b_l > \sum_{l=1}^{L} P_{b_l} + \xi \right) \) and the lemma can be proved by proving

\[ \lim_{T \to \infty} \Pr \left( \sum_{l=1}^{L} b_l > \sum_{l=1}^{L} P_{b_l} + \xi \right) = 0. \]

We start by observing that

\[ \Pr \left( \sum_{l=1}^{L} b_l > \sum_{l=1}^{L} P_{b_l} + \xi \right) \leq \Pr \left( \bigcup_{l=1}^{L} \left( \bar{b}_l > P_{b_l} + \frac{\xi}{L} \right) \right) \]

\[ \leq \sum_{l=1}^{L} \Pr \left( \bar{b}_l > P_{b_l} + \frac{\xi}{L} \right) \]

\[ = S_1 + S_2, \]

where \( S_1 \triangleq \sum_{l=\left\lfloor \frac{L}{3} \right\rfloor}^{L} \Pr \left( \bar{b}_l > P_{b_l} + \frac{\xi}{L} \right) \) and \( S_2 \triangleq \sum_{l=\left\lceil \frac{L}{3} \right\rceil + 1}^{L} \Pr \left( \bar{b}_l > P_{b_l} + \frac{\xi}{L} \right). \)

Starting with \( S_1 \), by the definition \( \bar{b}_l \) in (13), the \( l \)’th summand of \( S_1 \) is:

\[ \Pr \left( \bar{b}_l > P_{b_l} + \frac{\xi}{L} \right) = \Pr \left( \sum_{m=1}^{L} b_l^A(m) > k_{l-1}(P_{b_l} + \frac{\xi}{L}) \right). \] (32)

We recall that \( b_l^A(m) \) are Bernoulli(\( P_{b_l} \)) r.v.’s with limited independence. The following straightforward generalization of the Chernoff-Hoeffding Theorem is now useful:

**Lemma 11.** Let \( X_1, ..., X_n \) be a series of Bernoulli(\( p \)) r.v.’s, divided into groups of \( \ell \) elements. We assume that all distinct groups statistically independent but the r.v.’s within every group might be statistically dependent. Namely, let \( i, \tilde{i} \in \{1, ..., n/\ell\} \) and \( j, \tilde{j} \in \{1, ..., \ell\} \). It is given that \( X_{(i-1)\ell+j} \) and \( X_{(\tilde{i}-1)\ell+\tilde{j}\ell} \)
are statistically independent for every \( i \neq \tilde{i} \) and every \( j, \tilde{j} \) but might be statistically dependent for \( i = \tilde{i} \) and some \( j \neq \tilde{j} \). Then, for every \( 0 < \epsilon < 1 - p \):

\[
\Pr \left( \sum_{i=1}^{n} X_i \geq n(p + \epsilon) \right) \leq e^{-\frac{n \epsilon^2}{2}}. \tag{33}
\]

**Proof.** We begin with the standard derivation of the Chernoff bound for \( \sum_{i=1}^{n} X_i \):

\[
\Pr \left( \sum_{i=1}^{n} X_i \geq n(p + \epsilon) \right) \leq \min_{t > 0} e^{-tn(p + \epsilon)} \mathbb{E} \left( e^{t \sum_{i=1}^{n} X_i} \right) \\
\leq \min_{t > 0} e^{-tn(p + \epsilon)} \mathbb{E} \left( e^{t \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} X_{(i-1)\ell+j}} \right) \\
= \min_{t > 0} e^{-tn(p + \epsilon)} \prod_{i=1}^{n/\ell} \prod_{j=1}^{\ell} \mathbb{E} \left( e^{t X_{(i-1)\ell+j}} \right) \\
= \min_{t > 0} e^{-tn(p + \epsilon)} \prod_{i=1}^{n/\ell} \mathbb{E} \left( \prod_{j=1}^{\ell} e^{t X_{(i-1)\ell+j}} \right) \tag{34}
\]

where in (34) we used the independence assumptions of groups of length \( \ell \). We now prove the following bound for the first group, \( i = 1 \)

\[
\mathbb{E} \left( \prod_{j=1}^{\ell} e^{t X_j} \right) \leq \mathbb{E} \left( e^{t X_1} \right). \tag{35}
\]

The proof is based on using Hölder’s inequality iteratively. We start by recalling Hölder’s inequality for the expectation of real valued non-negative random variables, \( W, V \in \mathbb{R}, W, V \geq 0 \) and \( p > 1 \):

\[
\mathbb{E}(W \cdot V) \leq \left( \mathbb{E} \left( W^p \right) \right)^{\frac{1}{p}} \left( \mathbb{E} \left( V^q \right) \right)^{\frac{1}{q}}. \tag{36}
\]

Using (36) for \( \mathbb{E} \left( \prod_{j=1}^{\ell} e^{t X_j} \right) \) with \( W = \prod_{j=1}^{\ell-1} e^{t X_j}, V = e^{t X_1} \) and \( p = \ell \) gives

\[
\mathbb{E} \left( \prod_{j=1}^{\ell} e^{t X_j} \right) \leq \left( \mathbb{E} \prod_{j=1}^{\ell-1} e^{t X_j} \right)^{\frac{\ell-1}{\ell}} \left( \mathbb{E} \left( e^{t X_1} \right) \right)^{\frac{1}{\ell}}. \tag{37}
\]

Using (36) for \( \mathbb{E} \left( \prod_{j=1}^{\ell-1} e^{\frac{t}{\ell-1} X_j} \right) \) with \( W = \prod_{j=1}^{\ell-2} e^{\frac{t}{\ell-1} X_j}, V = e^{\frac{t}{\ell-1} X_{\ell-1}} \) and \( p = \ell - 1 \) gives

\[
\mathbb{E} \left( \prod_{j=1}^{\ell-1} e^{\frac{t}{\ell-1} X_j} \right) \leq \left( \mathbb{E} \prod_{j=1}^{\ell-2} e^{\frac{t}{\ell-1} X_j} \right)^{\frac{\ell-2}{\ell-1}} \left( \mathbb{E} \left( e^{\frac{t}{\ell-1} X_{\ell-1}} \right) \right)^{\frac{1}{\ell-1}}. \tag{38}
\]

Plugging (38) into (37) and taking into account that \( X_\ell \) and \( X_{\ell-1} \) have the same marginal distribution as \( X_1 \) gives:

\[
\mathbb{E} \left( \prod_{j=1}^{\ell} e^{t X_j} \right) \leq \left( \mathbb{E} \prod_{j=1}^{\ell-2} e^{\frac{t}{\ell-1} X_j} \right)^{\frac{\ell-2}{\ell-1}} \left( \mathbb{E} \left( e^{t X_1} \right) \right)^{\frac{2}{\ell-1}}. \tag{39}
\]
We now implement this process iteratively on the left hand term the upper bound in (39) for \( p = \ell - 2 \) to \( p = 2 \) finally giving (45).

We now notice that (35) depends only on the marginal distribution of a single sample, which is assumed to be Bernoulli(\( p \)), so it should hold for all groups \( i \in \{1, \ldots, n/\ell\} \). Therefore we can use (35) for all the elements in the outer product in (34) giving:

\[
\Pr \left( \sum_{i=1}^{n} X_i \geq n(p+\epsilon) \right) \leq \min_{t>0} e^{-tn(p+\epsilon)} \left( E \left( e^{tX_i} \right) \right)^{n/\ell} \\
\leq \left( \min_{t>0} e^{-t\ell(p+\epsilon)} E \left( e^{t\ell X_1} \right) \right)^{n/\ell} \\
e^{-\frac{\ell}{2} d_n((p+\epsilon)p).}
\]

where (40) is by the standard minimization of the Chernoff bound and \( d_n(p||q) \triangleq p \ln \frac{p}{q} + (1-p) \ln \frac{1-p}{1-q} \) is Kullback-Leibler Divergence between two Bernoulli random variable with probabilities \( p \) and \( q \), which is now calculated with respect to the natural logarithm basis. Finally, by Pinsker’s inequality we bound the divergence by \( d_n(p+\epsilon||p) \geq 2\epsilon^2 \) and obtain (33).

We can now use Lemma \( \text{II} \) to bound (32). Recalling the discussion from Subsection IX-A, at every layer \( 1 < l \leq L \), there are \( k^{L-l} \) blocks for which error detection is applied using Definition \( \text{III} \). We assume that only \( k^{(L-l)/2} \) independent test points are used, which are changed every \( k^{(L-l)/2} \) blocks. So, we can use Lemma \( \text{II} \) on (32) where the number of independent groups is \( \frac{n}{\ell} = k^{(L-l)/2} \) yielding:

\[
\Pr \left( \sum_{m=1}^{k^{L-l}} b^A_l(m) > k^{L-l}(P_{b_l} + \frac{\epsilon}{L}) \right) \leq e^{-k^{(L-l)/2}2\epsilon^2 L^2} \leq e^{-k^{(L-l)/2}2\epsilon^2 L^2}
\]

Summing all the element is of \( S_1 \) yields:

\[
S_1 \leq \sum_{l=1}^{\left\lfloor \frac{3L}{4} \right\rfloor} e^{-k^{(L-l)/2}2\epsilon^2 L^2} \leq \frac{3}{4} L \cdot e^{-k^{L/2}2\epsilon^2 L^2}
\]

The second transition is by using the maximal summand obtained at \( l = \left\lfloor \frac{3L}{4} \right\rfloor \). Recalling that \( L = \log_k T \), it is clear that \( \lim_{T \to \infty} S_1 = \lim_{L \to \infty} S_1 = 0 \).

Proceeding with \( S_2 \):

\[
S_2 = \sum_{l=\left\lfloor \frac{L}{4} \right\rfloor + 1}^{L} \Pr \left( \overline{b_l} > P_{b_l} + \frac{\epsilon}{L} \right) \\
\leq \sum_{l=\left\lfloor \frac{L}{4} \right\rfloor + 1}^{L} \Pr \left( \overline{b_l} > 0 \right)
\]

Observe that if \( \overline{b_l} > 0 \) then at least one rewind bit at level \( l \) is set to one. So, we can use the union bound and obtain

\[
\Pr \left( \overline{b_l} > 0 \right) \leq k^{L-l} P_{b_l}.
\]

Recalling (30)
we can further bound (13) by

\[ P_b \leq k^{2-l} \left( kP_{e1} + 3\beta^{a+4}k^2 \log k \frac{2 - \beta^2 k}{(1 - \beta^2 k)^2} \right) + 3\beta^a(\log k)l\beta^{2l} \]

Observing that the bound in (13) is monotonically decreasing in \( l \) for a sufficiently large \( l \) we can bound the summands of \( S_2 \) by the term obtained at \( l = 3/4L \), yielding:

\[ S_2 \leq \frac{L}{4}k^{-L/2} \left( k^3P_{e1} + 3\beta^{a+4}k^4 \log k \frac{2 - \beta^2 k}{(1 - \beta^2 k)^2} \right) + \frac{9}{16}\beta^a(\log k)L^2 \left( \left( \frac{\beta^2 k}{\epsilon} \right)^{3/4} \right)^L. \]

It is clear that the left hand term is monotonically decreasing in \( L \). Analyzing the right hand term, we use the definition of \( \beta \) and we observe that

\[ \left( \frac{\beta^2 k}{\epsilon} \right)^{3/4} \leq (\beta^6 k)^{1/4} < (2^6/(8k)^3 k)^{1/4} < (\epsilon k)^{-1/4} < 1 \]

where the third transition is due to the assumption that \( \epsilon < 1/(8k) \) in Theorem 3. All in all, setting \( L = \log k T \) guarantees that \( \lim_{T \to \infty} S_2 = 0 \), which concludes the proof of Lemma 6.

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