Low Complexity Schemes for the Random Access Gaussian Channel

Or Ordentlich (Hebrew University of Jerusalem)
Joint work with Yury Polyanskiy (MIT)

NSF Workshop on Low-Latency Wireless Random-Access,
MIT,
November 2, 2017
Massive Connectivity

Key challenge for next generation wireless networks:

Providing multiple-access to massive number of infrequently UNCOORDINATED communicating devices
Massive Connectivity

**Key challenge for next generation wireless networks:**

Providing multiple-access to massive number of infrequently **UNCOORDINATED** communicating devices

Typical scenario:

- Very large number of users, say $K_{tot} \approx 10^6 – 10^7$
- Much smaller, but still large, number of them active within each communication block, say $K_a \approx 50 – 500$
- Message of each active user is short, say $k = 100$ bits
- Blocklength is $n \approx 10^4 – 10^5$
- $\frac{k}{n} \ll 1$, but the total spectral efficiency $\rho = \frac{K_a \cdot k}{n}$ is moderate, say 1 bit per channel use
Massive Connectivity

Key challenge for next generation wireless networks:

Providing multiple-access to massive number of infrequently UNCOORDINATED communicating devices

Typical scenario:

- Very large number of users, say $K_{tot} \approx 10^6 - 10^7$
- Much smaller, but still large, number of them active within each communication block, say $K_a \approx 50 - 500$
- Message of each active user is short, say $k = 100$ bits
- Blocklength is $n \approx 10^4 - 10^5$
- $\frac{k}{n} \ll 1$, but the total spectral efficiency $\rho = \frac{K_a \cdot k}{n}$ is moderate, say 1 bit per channel use

The goal is to communicate with the smallest possible energy-per-bit
Gaussian Random Access Channel

\[ y = \sum_{i=1}^{K_{\text{tot}}} s_i x_i + z, \]

- \( K_{\text{tot}} \) possible users, \( K_a \) of them are active
- \((s_1, \ldots, s_{K_{\text{tot}}}) \in \{0, 1\}^{K_{\text{tot}}} \) with Hamming weight \( K_a \), unknown
- \( z \sim \mathcal{N}(0, I) \), \( x_i \in \mathbb{R}^n \), \( \|x_i\|^2 \leq nP \)
- Each active user has a \( k \)-bit message \( M_i \)
Gaussian Random Access Channel

\[ y = \sum_{i=1}^{K_{\text{tot}}} s_i x_i + z, \]

- \( K_{\text{tot}} \) possible users, \( K_a \) of them are active
- \((s_1, \ldots, s_{K_{\text{tot}}}) \in \{0, 1\}^{K_{\text{tot}}} \) with Hamming weight \( K_a \), unknown
- \( z \sim \mathcal{N}(0, I) \), \( x_i \in \mathbb{R}^n, \|x_i\|^2 \leq nP \)
- Each active user has a \( k \)-bit message \( M_i \)

Decoder’s job: Output an unordered list \( \mathcal{L}(y) \) of \( K_a \) messages that contains most messages that were transmitted

user identification handled in higher layers
Gaussian Random Access Channel

\[ y = \sum_{i=1}^{K_{\text{tot}}} s_i x_i + z, \]

- \( K_{\text{tot}} \) possible users, \( K_a \) of them are active
- \( (s_1, \ldots, s_{K_{\text{tot}}}) \in \{0, 1\}^{K_{\text{tot}}} \) with Hamming weight \( K_a \), unknown
- \( z \sim \mathcal{N}(0, I) \), \( x_i \in \mathbb{R}^n, \left\| x_i \right\|^2 \leq nP \)
- Each active user has a \( k \)-bit message \( M_i \)

**Decoder’s job:** Output an unordered list \( \mathcal{L}(y) \) of \( K_a \) messages that contains most messages that were transmitted

user identification handled in higher layers

\[ P_e = \max_{\left( s_1, \ldots, s_{K_{\text{tot}}} \right) = K_a} \frac{1}{K_a} \sum_{i=1}^{K_{\text{tot}}} s_i \cdot \Pr \left( M_i \notin \mathcal{L}(y) \right) \]
Gaussian Random Access Channel

\[ y = \sum_{i=1}^{K_{\text{tot}}} s_i x_i + z, \]

- \( K_{\text{tot}} \) possible users, \( K_a \) of them are active
- \( (s_1, \ldots, s_{K_{\text{tot}}}) \in \{0,1\}^{K_{\text{tot}}} \) with Hamming weight \( K_a \), unknown
- \( z \sim \mathcal{N}(0,I) \), \( x_i \in \mathbb{R}^n, \|x_i\|^2 \leq nP \)
- Each active user has a \( k \)-bit message \( M_i \)

Decoder’s job: Output an unordered list \( \mathcal{L}(y) \) of \( K_a \) messages that contains most messages that were transmitted

\[ P_e = \max_{|\{s_1, \ldots, s_{K_{\text{tot}}}\}| = K_a} \frac{1}{K_a} \sum_{i=1}^{K_{\text{tot}}} s_i \cdot \Pr(M_i \notin \mathcal{L}(y)) \]

Formulation does not become trivial for \( K_{\text{tot}} = \infty \)
Gaussian Random Access Channel

\[ y = \sum_{i=1}^{K_{\text{tot}}} s_i x_i + z, \]

- \( K_{\text{tot}} \) possible users, \( K_a \) of them are active
- \( (s_1, \ldots, s_{K_{\text{tot}}}) \in \{0,1\}^{K_{\text{tot}}} \) with Hamming weight \( K_a \), unknown
- \( z \sim \mathcal{N}(0, I) \), \( x_i \in \mathbb{R}^n, \|x_i\|^2 \leq nP \)
- Each active user has a \( k \)-bit message \( M_i \)

Decoder’s job: Output an unordered list \( \mathcal{L}(y) \) of \( K_a \) messages that contains most messages that were transmitted

user identification handled in higher layers

\[ P_e = \max_{|(s_1, \ldots, s_{K_{\text{tot}}})| = K_a} \frac{1}{K_a} \sum_{i=1}^{K_{\text{tot}}} s_i \cdot \Pr(M_i \notin \mathcal{L}(y)) \]

Goal: For given \( (n, k, K_a, P_e) \) minimize \( \frac{E_b}{N_0} \triangleq \frac{nP}{2k} \)
Potential Coding Schemes

Natural approach is to use the same codebook for all transmitters
Potential Coding Schemes

Natural approach is to use the same codebook for all transmitters

Random coding achievability bound (Polyanskiy, ISIT’17)

$n = 30,000$, $k = 100$, $P_e = 0.05$
Potential Coding Schemes

What about more practical solutions?
Potential Coding Schemes

What about more practical solutions?

Option I - treat interference as noise (un-coordinated CDMA)
Total spectral efficiency is limited $\rho \triangleq \frac{K_a \cdot k}{n} < \frac{\log(e)}{2}$ bits channel use

\[ n = 30,000, \ k = 100, \ P_e = 0.05 \]
Potential Coding Schemes

What about more practical solutions?

Option II - slotted-ALOHA
Works well for $P_e \geq 1 - \frac{1}{e}$. Otherwise, to keep collision probability below $P_e$ the fraction of utilized slots is $\approx \ln \frac{1}{1-P_e}$
Potential Coding Schemes

Not quite practical, but getting closer...
Potential Coding Schemes

Not quite practical, but getting closer...

T-fold ALOHA : ALOHA with good channel codes for T-user MAC
Collisions of \( \leq T \) users can be resolved
\( \Rightarrow \) fraction of utilized slots increases

\[ n = 30,000, \quad k = 100, \quad P_e = 0.05 \]
Potential Coding Schemes

This work: a practical realization of this idea
Potential Coding Schemes

This work: a practical realization of this idea

Low-complexity (suboptimal) code design for the $T$-user MAC with same codebook for all users, combined with $T$-fold ALOHA

$n = 30,000$, $k = 100$, $P_e = 0.05$
High-Level Overview of Scheme

Split the $n$ channel uses to $V$ sub-blocks
High-Level Overview of Scheme

Split the $n$ channel uses to $V$ sub-blocks

All users encode their messages using the same codebook $C \in \mathbb{R}^{n/V}$
High-Level Overview of Scheme

Split the \( n \) channel uses to \( V \) sub-blocks

All users encode their messages using the same codebook \( C \in \mathbb{R}^{n/V} \)

Each active user maps its message to a codeword in \( C \), and transmits it over one sub-block chosen uniformly over \([V]\)
High-Level Overview of Scheme

Split the $n$ channel uses to $V$ sub-blocks

All users encode their messages using the same codebook $C \in \mathbb{R}^{n/V}$

Each active user maps its message to a codeword in $C$, and transmits it over one sub-block chosen uniformly over $[V]$

If $L > T$ users transmitted during a sub-block, all their messages are **not delivered**.

Otherwise, the decoder attempts to recover the $L$ messages and add them to its list
High-Level Overview of Scheme

Split the $n$ channel uses to $V$ sub-blocks

All users encode their messages using the same codebook $C \in \mathbb{R}^{n/V}$

Each active user maps its message to a codeword in $C$, and transmits it over one sub-block chosen uniformly over $[V]$.

If $L > T$ users transmitted during a sub-block, all their messages are not delivered.

Otherwise, the decoder attempts to recover the $L$ messages and add them to its list.

**Coding task:** design a codebook $C$ with efficient encoding/decoding algorithms and good performance for $T$-user MAC
High-Level Overview of Scheme

We will use a **concatenated code**:  
Inner code will deal with noise (*CoF phase*)
Outer code with interaction between codewords (*BAC phase*)
High-Level Overview of Scheme

Inner code $C_{\text{lin}}$ converts the $T$-user Gaussian MAC into a modulo-$p$ noiseless adder MAC.
This is done via compute-and-forward [Nazer-Gastpar’11]
High-Level Overview of Scheme

Inner code $C_{\text{lin}}$ converts the $T$-user Gaussian MAC into a modulo-$p$ noiseless adder MAC. This is done via compute-and-forward [Nazer-Gastpar’11]

$w_1, \ldots, w_T$ are vectors in $\mathbb{Z}_p$

$C_{\text{lin}}$ is linear code over $\mathbb{Z}_p$
High-Level Overview of Scheme

Inner code $C_{\text{lin}}$ converts the $T$-user Gaussian MAC into a modulo-$p$ noiseless adder MAC. This is done via compute-and-forward [Nazer-Gastpar’11]

\[ y_{\text{BAC}} = \left[ \sum_{i=1}^{T} w_i \right] \mod p \]

$w_1, \ldots, w_T$ are vectors in $\mathbb{Z}_p$  

$C_{\text{lin}}$ is linear code over $\mathbb{Z}_p$
High-Level Overview of Scheme

Outer code $C_{BAC}$ is designed for the $T$-user modulo-$p$ adder MAC

$w_1, \ldots, w_T$ are vectors in $\mathbb{Z}_p$

$C_{\text{lin}}$ is linear code over $\mathbb{Z}_p$

$$y_{BAC} = \left[ \sum_{i=1}^{T} w_i \right] \mod p$$
High-Level Overview of Scheme

The most practical choice is $p = 2$ and this is our focus.

$w_1, \ldots, w_T$ are vectors in $\mathbb{Z}_p$

$C_{\text{lin}}$ is linear code over $\mathbb{Z}_p$

$y_{\text{BAC}} = \left[ \sum_{i=1}^{T} w_i \right] \mod p$
Related Work

Most of the ideas we use appeared before in various contexts. Their combination to a coding scheme for the Gaussian random access channel is new.

- Compute-and-forward [Nazer-Gastpar’11]
- Explicit codes for the modulo-2 binary adder channel [Lindström’69, Bar-David et al.’93]
- Concatenation of codes with good minimum distance and codes for the BAC [Ericson-Levenshtein’94]
- Concatenation of CoF inner codes with syndrome decoding for compressed sensing [Lee-Hong’16]
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint

Receiver sees

$y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get

$y_{\text{CoF}} = \left[ x + z \right] \mod 2$

The channel from $x$ to $y_{\text{CoF}}$ is a BMS

Designing $C_{\text{lin}}$ is a standard coding task
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint.
- Receiver sees $y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get

$$y_{\text{CoF}} = [x + z] \mod 2$$

where $x = [\sum_i x_i] \mod 2 \in C_{\text{lin}}$
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint.
- Receiver sees $y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get
  \[
  y_{\text{CoF}} = [x + z] \mod 2
  \]
  where $x = [\sum_i x_i] \mod 2 \in C_{\text{lin}}$
- The channel from $x$ to $y_{\text{CoF}}$ is a BMS
  $\implies$ Designing $C_{\text{lin}}$ is a standard coding task
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint.
- Receiver sees $y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get

$$y_{\text{CoF}} = [x + z] \mod 2$$

where $x = [\sum_i x_i] \mod 2 \in C_{\text{lin}}$
- The channel from $x$ to $y_{\text{CoF}}$ is a BMS
  $\implies$ Designing $C_{\text{lin}}$ is a standard coding task

*What is lost in the conversion $y \mapsto y_{\text{CoF}}$?*
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint.
- Receiver sees $y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get
  \[ y_{\text{CoF}} = [x + z] \mod 2 \]
  where $x = [\sum_i x_i] \mod 2 \in C_{\text{lin}}$

- The channel from $x$ to $y_{\text{CoF}}$ is a BMS
  $\implies$ Designing $C_{\text{lin}}$ is a standard coding task

**What is lost in the conversion $y \leftrightarrow y_{\text{CoF}}$?**

- Sum-capacity of $y$ grows like $\log(T \cdot P)$
- Capacity of $y_{\text{CoF}}$ only grows like $\log(P)$
More on the CoF phase

- We construct $C_{\text{lin}}$ from a binary linear code, shifted and scaled to meet the power constraint.
- Receiver sees $y = \sum_{i=1}^{T} x_i + z$, after shifting and scaling $y$, and reducing modulo-2, we get

$$y_{\text{CoF}} = [x + z] \mod 2$$

where $x = \left[\sum_i x_i\right] \mod 2 \in C_{\text{lin}}$

- The channel from $x$ to $y_{\text{CoF}}$ is a BMS.

$\implies$ Designing $C_{\text{lin}}$ is a standard coding task.

What is lost in the conversion $y \mapsto y_{\text{CoF}}$?

Sum-capacity of $y$ grows like $\log(T \cdot P)$
Capacity of $y_{\text{CoF}}$ only grows like $\log(P)$

$T$-fold ALOHA reduces “power-loss” to $1/T$ instead of $1/K_a$
More on the BAC Phase

\[ y_{\text{BAC}} = \left[ \sum_{i=1}^{T} w_i \right] \mod 2, \quad w_1, \ldots, w_T \in C_{\text{BAC}} \]

Need to decode a list \( \{w_1, \ldots, w_T\} \)
More on the BAC Phase

\[ y_{BAC} = \left[ \sum_{i=1}^{T} w_i \right] \mod 2, \quad w_1, \ldots, w_T \in C_{BAC} \]

Need to decode a list \( \{w_1, \ldots, w_T\} \)

The symmetric-capacity of this MAC is \( 1/T \) bits/channel use
More on the BAC Phase

\[ y_{\text{BAC}} = \left[ \sum_{i=1}^{T} w_i \right] \mod 2, \quad w_1, \ldots, w_T \in C_{\text{BAC}} \]

How to construct explicit codes?

- Let \( H = [h_1 | \cdots | h_n] \) be the parity-check matrix of a \([n, k]\) binary \( T \)-error correcting code.
- All linear combinations of at most \( T \) columns are distinct.
- The code \( C_{\text{BAC}} = \{h_1, \ldots, h_n\} \) is (almost) zero-error for the \( T \)-user mod-2 adder MAC.
- \( R_{\text{BAC}} = \frac{\log n}{n-k} \)
More on the BAC Phase

\[ \mathbf{y}_{\text{BAC}} = \left[ \sum_{i=1}^{T} \mathbf{w}_i \right] \mod 2, \quad \mathbf{w}_1, \ldots, \mathbf{w}_T \in \mathcal{C}_{\text{BAC}} \]

How to construct explicit codes?

- Let \( H = [\mathbf{h}_1 | \cdots | \mathbf{h}_n] \) be the parity-check matrix of a \([n, k]\) binary \( T\)-error correcting code
- All linear combinations of at most \( T \) columns are distinct
- The code \( \mathcal{C}_{\text{BAC}} = \{\mathbf{h}_1, \ldots, \mathbf{h}_n\} \) is (almost) zero-error for the \( T\)-user mod-2 adder MAC
- \( R_{\text{BAC}} = \frac{\log n}{n-k} \)

\( \forall r \geq 3, \exists [n = 2^r - 1, n - k = rT, d \geq 2T + 1] \) binary BCH code

Rate of induced \( \mathcal{C}_{\text{BAC}} \) is \( R_{\text{BAC}} = \frac{\log 2^r - 1}{rT} \approx \frac{1}{T} \)
More on the BAC Phase

\[ \mathbf{y}_{\text{BAC}} = \left[ \sum_{i=1}^{T} \mathbf{w}_i \right] \mod 2, \quad \mathbf{w}_1, \ldots, \mathbf{w}_T \in \mathcal{C}_{\text{BAC}} \]

How to construct explicit codes?

- Let \( H = [\mathbf{h}_1 | \cdots | \mathbf{h}_n] \) be the parity-check matrix of a \([n, k]\)
  binary \( T \)-error correcting code
- All linear combinations of at most \( T \) columns are distinct
- The code \( \mathcal{C}_{\text{BAC}} = \{\mathbf{h}_1, \ldots, \mathbf{h}_n\} \) is (almost) zero-error for the \( T \)-user mod-2 adder MAC
- \( R_{\text{BAC}} = \frac{\log n}{n-k} \)

\( \forall r \geq 3, \exists [n = 2^r - 1, n - k = rT, d \geq 2T + 1] \) binary BCH code

Rate of induced \( \mathcal{C}_{\text{BAC}} \) is \( R_{\text{BAC}} = \frac{\log 2^r - 1}{rT} \approx \frac{1}{T} \)

---

**Problem:** decoding complexity of BCH linear in \( n = 2^r - 1 \)
More on the BAC Phase - Encoding & Decoding

Encoding:

Each column of $H$ is of the form $[\alpha, \alpha^3, \ldots, \alpha^{2T-1}]$, for $\alpha \in GF(2^k) \setminus \{0\}$

To encode, just map messages to elements of $GF(2^k) \setminus \{0\}$ and compute first $T$ odd powers
More on the BAC Phase - Encoding & Decoding

Decoding:

Modified GPZ algorithm, almost the same as Bar-David et al.’93

- **Syndrome computation**: odd syndromes given “for free” from \( y_{BAC} \). Computing even syndromes from them is easy

- **Construction of error locator polynomial**: Berlekamp-Massey algorithm gives

\[
\sigma(X) = 1 + \sum_{t=1}^{L} \sigma_t X^t = \prod_{i=1}^{T} (1 + \alpha_i X)
\]

where \( \alpha_1, \ldots, \alpha_T \) correspond to the messages

- **Finding the roots of \( \sigma(X) \)**: Rabin’s probabilistic algorithm [Rabin’80] finds \( \alpha_1^{-1}, \ldots, \alpha_T^{-1} \)

- **Inversion of the roots**: using the identity \( \alpha^{-1} = \alpha^{2^k} - 1 \)

Total complexity: \( \mathcal{O}(kT^2 \log^2(T) \log \log(T)) \) operations in \( GF(2^k) \)
Spectral Efficiency > 1

The spectral efficiency \( \rho = \frac{K_a \cdot k}{n} \) of our scheme is at most \( R_{\text{lin}} \)

What if \( \rho > 1 \)?

Option 1 - work with \( p > 2 \)

- CoF phase requires good linear codes over \( \mathbb{F}_p \)
- BAC phase can be implemented using \( H = [h_1 | \cdots | h_n] \) of a \([n = p^s - 1, n - k = 2T]\) Reed-Solomon code over \( \mathbb{F}_{p^s} \) with

\[
C_{\text{BAC}} = \{ \alpha h_i : \alpha \in \mathbb{F}_{p^s} \setminus \{0\}, i = 1, \ldots, p^s - 1 \}
\]

- Can also use a nested lattice code to achieve the 1.53dB shaping gain
Spectral Efficiency > 1

The spectral efficiency $\rho = \frac{K_a \cdot k}{n}$ of our scheme is at most $R_{\text{lin}}$

What if $\rho > 1$?

Option I - work with $p > 2$

- CoF phase requires good linear codes over $\mathbb{F}_p$
- BAC phase can be implemented using $H = [h_1 | \cdots | h_n]$ of a $[n = p^s - 1, n - k = 2T]$ Reed-Solomon code over $\mathbb{F}_{p^s}$ with

$$C_{\text{BAC}} = \{\alpha h_i : \alpha \in \mathbb{F}_{p^s} \setminus \{0\}, i = 1, \ldots, p^s - 1\}$$

- Can also use a nested lattice code to achieve the 1.53dB shaping gain

But encoders in our setup must be extremely simple

$\implies$ binary codes are preferable
Spectral Efficiency > 1

The spectral efficiency $\rho = \frac{K_a \cdot k}{n}$ of our scheme is at most $R_{\text{lin}}$.

What if $\rho > 1$?

Option II - use a multilevel code based on binary codes

- Allows to increase $R_{\text{lin}}$ above 1
- Requires some overhead in order to “pair” messages from different layers
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $\left(\frac{E_b}{N_0}\right)^* = \frac{2^{2\rho}-1}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $\left(\frac{E_b}{N_0}\right)^*$?
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $\left( \frac{E_b}{N_0} \right)^* = \frac{2^{2\rho - 1}}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $\left( \frac{E_b}{N_0} \right)^*$?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$. 
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $\left( \frac{E_b}{N_0} \right)^* = \frac{2^{2\rho} - 1}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $\left( \frac{E_b}{N_0} \right)^*$?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$. The $T$-collision probability is $\Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right)$.
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $\left( \frac{E_b}{N_0} \right)^* = 2^{2\rho - 1} \frac{1}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $\left( \frac{E_b}{N_0} \right)^*$?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$. The $T$-collision probability is $\Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right)$

Linear code required to have rate $R_{\text{lin}} = \frac{\rho}{\alpha}$
Approximate performance

A TDMA scheme with infinite blocklength and fixed \( K_a \) can achieve \( \left( \frac{E_b}{N_0} \right)^* = \frac{2^{2\rho} - 1}{2\rho} \) where \( \rho = \frac{K_a \cdot k}{n} \).

How far are we from \( \left( \frac{E_b}{N_0} \right)^* \)?

Fix \( T \) and set the number of sub-blocks to \( V = \frac{K_a}{\alpha T} \) for \( \alpha \in (0, 1] \).

The \( T \)-collision probability is \( \Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right) \)

Linear code required to have rate \( R_{\text{lin}} = \frac{\rho}{\alpha} \)

\[ \Delta = \left( \frac{E_b}{N_0} \right)_{\text{dB}} - \left( \frac{E_b}{N_0} \right)^*_{\text{dB}} \]
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve \( \left( \frac{E_b}{N_0} \right)^* = \frac{2^{2\rho} - 1}{2\rho} \) where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from \( \left( \frac{E_b}{N_0} \right)^* \)?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$.

The $T$-collision probability is $\Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right)$

Linear code required to have rate $R_{\text{lin}} = \frac{\rho}{\alpha}$

\[
\Delta = \left( \frac{E_b}{N_0} \right) \text{dB} - \left( \frac{E_b}{N_0} \right)^* \text{dB}
\]

\[
\approx 6\rho \frac{1 - \alpha}{\alpha} + 10 \log_{10}(\alpha)
\]

T-Collision avoidance loss due to a $1/\alpha$ increase in spectral efficiency
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $\left(\frac{E_b}{N_0}\right)^* = \frac{2^{2\rho-1}}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $\left(\frac{E_b}{N_0}\right)^*$?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$.

The $T$-collision probability is $\Pr(\text{Binomial}(K_a - 1, \frac{\alpha T}{K_a}) \geq T)$

Linear code required to have rate $R_{\text{lin}} = \frac{\rho}{\alpha}$

$$\Delta = \left(\frac{E_b}{N_0}\right) \text{dB} - \left(\frac{E_b}{N_0}\right)^* \text{dB}$$

$$\approx 6\rho \frac{1 - \alpha}{\alpha} + 10 \log_{10}(\alpha) + 10 \log_{10}(T)$$

CoF loss from the reduction $y \mapsto y_{\text{CoF}}$
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve \( \left( \frac{E_b}{N_0} \right)^* = \frac{2^{2\rho} - 1}{2\rho} \) where \( \rho = \frac{K_a \cdot k}{n} \).

How far are we from \( \left( \frac{E_b}{N_0} \right)^* \)?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$.

The $T$-collision probability is $\Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right)$.

Linear code required to have rate $R_{\text{lin}} = \frac{\rho}{\alpha}$

\[
\Delta = \left( \frac{E_b}{N_0} \right) \text{dB} - \left( \frac{E_b}{N_0} \right)^* \text{dB}
\]

\[
\approx 6\rho \frac{1 - \alpha}{\alpha} + 10 \log_{10}(\alpha) + 10 \log_{10}(T) - 10 \log_{10}(1 - 2^{-2\rho})
\]

Loss of $+1$ in computation rate
Approximate performance

A TDMA scheme with infinite blocklength and fixed $K_a$ can achieve $(\frac{E_b}{N_0})^* = \frac{2^{2\rho} - 1}{2\rho}$ where $\rho = \frac{K_a \cdot k}{n}$.

How far are we from $(\frac{E_b}{N_0})^*$?

Fix $T$ and set the number of sub-blocks to $V = \frac{K_a}{\alpha T}$ for $\alpha \in (0, 1]$.

The $T$-collision probability is $\Pr \left( \text{Binomial} \left( K_a - 1, \frac{\alpha T}{K_a} \right) \geq T \right)$.

Linear code required to have rate $R_{\text{lin}} = \frac{\rho}{\alpha}$

$$\Delta = \left( \frac{E_b}{N_0} \right) \text{ dB} - \left( \frac{E_b}{N_0} \right)^* \text{ dB}$$

$$\approx 6\rho \frac{1 - \alpha}{\alpha} + 10 \log_{10}(\alpha) + 10 \log_{10}(T) - 10 \log_{10}(1 - 2^{-2\rho}) + 1.53$$

Shaping loss
Approximate performance
Summary and Conclusions

- We considered T-fold ALOHA (MPR) as a candidate coding scheme for the Gaussian random access channel.
- We constructed a practical variant of T-fold ALOHA based on concatenation of CoF codes and BAC codes.
- Our scheme is far from optimal, but significantly outperforms competing low-complexity schemes in regimes of practical interest.